International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2009.**

1. In a convex 2009-gon, all diagonals are drawn. A line intersects the 2009-gon but does not pass through any of its vertices. Prove that the line intersects an even number of diagonals.

2. Let $a \wedge b$ denote the number a^b. The order of operations in the expresson $7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7$ must be determined by inserting five pairs of brackets. Is it possible to put brackets in two distinct ways so that the expressions have the same value?

3. Vlad is going to print a digit on each face of several unit cubes, in such a way that a 6 does not turn into a 9. If it is possible to form every 30-digit number with these blocks, what is the minimum number of the blocks?

4. When a positive integer is increased by 10%, the result is another positive integer whose digit-sum has decreased by 10%. Is this possible?

5. In the rhombus $ABCD$, $\angle A = 120°$. M is a point on BC and N is a point on CD such that $\angle MAN = 30°$. Prove that the circumcentre of triangle MAN lies on a diagonal of $ABCD$.

Note: The problems are worth 3, 4, 4, 4 and 5 points respectively.

Courtesy of Andy Liu

Solution to Junior O-Level Spring 2009

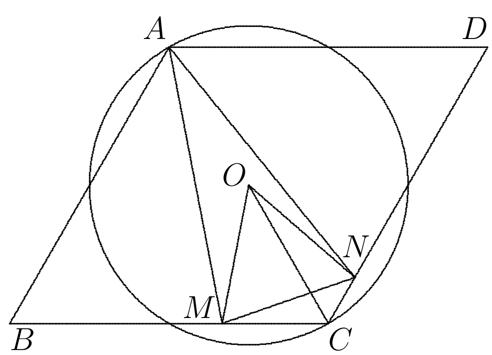
1. Let there be m vertices of the convex 2009-gon on one side of the line and n vertices on the other side. Since $m+n=2009$, one of m and n is odd and the other is even. Hence mn is even. The line intersects mn segments which join two points on opposite sides. Two of them are sides of the 2009-gon, but the remaining $mn-2$ are diagonals, and this number is even.

2. **Solution by Olga Ivrii:**
More generally, $(n\wedge(n\wedge n))\wedge n=(n^{n\wedge n})^n=(n^n)^{n\wedge n}=(n\wedge n)\wedge(n\wedge n)$. Adding three more terms to both sides the same way maintains the equal value.

3. **Solution by Olga Ivrii:**
We need at least 30 copies of each non-zero digits because there is a 30-digit number consisting only of that digit. We need at least 29 copies of zero because there is a 30-digit number whose last 29 digits are zeros. Since $30\times 10=50\times 6$, Vlad needs at least 50 cubes. His first five cubes may consist of the numbers (0,1,2,3,4,5), (6,7,8,9,0,1), (2,3,4,5,6,7), (8,9,0,1,2,3) and (4,5,6,7,8,9). Since each digit appears three times and no two copies of the same number appear on the same cube, Vlad can use this set to form any 3-digit number. If he makes nine more copies of this set, he can use the 50 cubes to form any 30-digit number.

4. **Solution by Olga Ivrii:**
It is possible, and a lot of carrying is involved in changing the old number to the new. We want the old number to start with a block of 9s, and we want its last digit to be 0. However, a number with a lone 0 following a block of 9s has the same digit-sum when multiplied by $\frac{11}{10}$. So we put m 9s in front, a lone 0 at the end, and n 5s in between. When this number is multiplied by $\frac{11}{10}$, the new number consists of 10, $m-1$ 9s, 5, $n-1$ 1s and 05 in that order. The digit-sum of the original number is $9m+5n$ while the digit-sum of the new number is $9m+n$. From $9(9m+5n)=10(9m+n)$, we have $35n=9m$. Hence we can choose $m=35$ and $n=9$.

5. **Solution by Olga Ivrii:**



Let O be the circumcentre of MAN. Then $\angle MON=2\angle MAN=60°$. Hence OMN is an equilateral triangle. Since $\angle MON+\angle MCN=180°$, $CMON$ is a cyclic quadrilateral. Now $\angle OCM=\angle ONM=60°=\angle OMN=\angle OCN$. Hence O lies on the bisector of $\angle MCN$, which is the diagonal AC.