International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2009.**[1]

1 [**3**] There are two numbers on a board, 1/2009 and 1/2008. Alex and Ben play the following game. At each move, Alex names a number x (of his choice), while Ben responds by increasing one of the numbers on the board (of his choice) by x. Alex wins if at some moment one of the numbers on the board becomes 1. Can Alex win (no matter how Ben plays)?

2 (**a**) [**2**] Find a polygon which can be cut by a straight line into two congruent parts so that one side of the polygon is divided in half while another side at a ratio of $1:2$.

(**b**) [**3**] Does there exist a convex polygon with this property?

3 [**5**] In each square of a 101×101 board, except the central one, is placed either a sign "turn" or a sign "straight". The chess piece "car" can enter any square on the boundary of the board from outside (perpendicularly to the boundary). If the car enters a square with the sign "straight" then it moves to the next square in the same direction, otherwise (in case it enters a square with the sign "turn") it turns either to the right or to the left (its choice). Can one place the signs in such a way that the car never enter the central square?

4 [**5**] Consider an infinite sequence consisting of distinct positive integers such that each term (except the first one) is either an arithmetic mean or a geometric mean of two neighboring terms. Does it necessarily imply that starting at some point the sequence becomes either arithmetic progression or a geometric progression?

5 [**6**] A castle is surrounded by a circular wall with 9 towers which are guarded by knights during the night. Every hour the castle clock strikes and the guards shift to the neighboring towers; each guard always moves in the same direction (either clockwise or counterclockwise). Given that (i) during the night each knight guards every tower (ii) at some hour each tower was guarded by at least two knights (iii) at some hour exactly 5 towers were guarded by single knights, prove that at some hour one of the towers was unguarded.

6 [**7**] Angle C of an isosceles triangle ABC equals 120°. Each of two rays emitting from vertex C (inwards the triangle) meets AB at some point (P_i) reflects according to the rule "the angle of incidence equals the angle of reflection" and meets lateral side of triangle ABC at point Q_i ($i = 1, 2$). Given that angle between the rays equals 60°, prove that area of triangle P_1CP_2 equals the sum of areas of triangles AQ_1P_1 and BQ_2P_2 ($AP_1 < AP_2$).

7 [**9**] Let $\binom{n}{k}$ be the number of ways that k objects can be chosen (regardless of order) from a set of n objects. Prove that if positive integers k and l are greater than 1 and less than n, then integers $\binom{n}{k}$ and $\binom{n}{l}$ have a common divisor greater than 1.

[1] Courtesy of Andy Liu