International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2009.**[1]

1 [**4**] One hundred pirates played cards. When the game was over, each pirate calculated the amount he won or lost. The pirates have a gold sand as a currency; each has enough to pay his debt.

Gold could only change hands in the following way. Either one pirate pays an equal amount to every other pirate, or one pirate receives the same amount from every other pirate.

Prove that after several such steps, it is possible for each winner to receive exactly what he has won and for each loser to pay exactly what he has lost.

2 [**6**] A non-square rectangle is cut into N rectangles of various shapes and sizes. Prove that one can always cut each of these rectangles into two rectangles so that one can construct a square and rectangle, each figure consisting of N pieces.

3 [**7**] Every edge of a tetrahedron is tangent to a given sphere. Prove that the three line segments joining the points of tangency of the three pairs of opposite edges of the tetrahedron are concurrent.

4 [**8**] Denote by $[n]!$ the product $1 \cdot 11 \cdot \ldots \cdot \underbrace{11\ldots1}_{n \text{ ones}}$ (n factors in total). Prove that $[n+m]!$ is divisible by $[n]! \times [m]!$.

5 [**8**] Let XYZ be a triangle. The convex hexagon $ABCDEF$ is such that AB, CD and EF are parallel and equal to XY, YZ and ZX, respectively. Prove that area of triangle with vertices at the midpoints of BC, DE and FA is no less than area of triangle XYZ.

6 [**10**] Anna and Ben decided to visit Archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben are playing during the trip:

Anna chooses the first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna could always win, regardless of the way Ben played and regardless of the way the islands were connected.

7 [**11**] At the entrance to a cave is a rotating round table. On top of the table are n identical barrels, evenly spaced along its circumference. Inside each barrel is a herring either with its head up or its head down. In a move, Ali Baba chooses from 1 to n of the barrels and turns them upside down. Then the table spins around. When it stops, it is impossible to tell which barrels have been turned over. The cave will open if the heads of the herrings in all n barrels are up or are all down. Determine all values of n for which Ali Baba can open the cave in a finite number of moves.

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[1]Courtesy of Andy Liu