International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2008.**

1. There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

2. Can it happen that the least common multiple of 1, 2, $\ldots$, n is 2008 times the least common multiple of 1, 2, $\ldots$, m for some positive integers m and n?

3. In triangle ABC, $\angle A = 90°$. M is the midpoint of BC and H is the foot of the altitude from A to BC. The line passing through M and perpendicular to AC meets the circumcircle of triangle AMC again at P. If BP intersects AH at K, prove that $AK = KH$.

4. No matter how two copies of a convex polygon are placed inside a square, they always have a common point. Prove that no matter how three copies of the same polygon are placed inside this square, they also have a common point.

5. We may permute the rows and the columns of the table below. How may different tables can we generate?

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

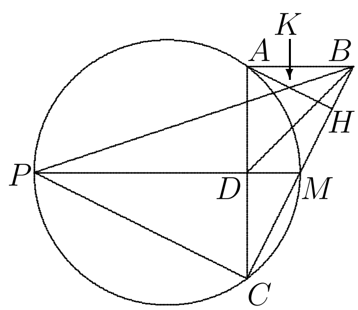
Note: The problems are worth 4, 5, 5, 5 and 6 points respectively.

Solution to Senior O-Level Spring 2008

1. Suppose none of a, b and c is 0. They cannot all be positive and they cannot be all negative. By symmetry, we may assume that a and b are positive while c is negative. Since a and b are distinct, we may assume that $a > b$. If $a > -c$, we take five cards with a on each. Then it is impossible to take another five cards to bring the total down to 0. If $-c > a$, we take five cards with c on each. Then it is impossible to take another five cards to bring the total up to 0. It follows that we must have $a = -c > b$. If we now take four cards with a on each and a fifth card with b on it, it is impossible to take another five cards to bring the total down to 0.

2. Let the highest power of 2 less than or equal to m be 2^r. Since $2008 = 2^3 \times 251$, the highest power of 2 less than or equal to n must be 2^{r+3}. It follows that $n > 4m$. Let the highest power of 3 less than or equal to m be 3^s. Then the highest power of 3 less than or equal to n must also be 3^s since 3 does not divide 2008. However, $n > 4m \geq 4 \times 3^s > 3^{s+1}$, which is a contradiction. Hence no such positive integers m and n exist.

3. Since both AB and MP are perpendicular to AC and $BM = MC$, MP intersects AC at its midpoint D. It follows that MP is a diameter of the circumcircle, so that MC is perpendicular to PC. It follows that triangles MCD and MPC are similar, so that $\frac{MD}{MC} = \frac{MC}{MP}$. Hence $\frac{MD}{MB} = \frac{MB}{MP}$. Since $\angle DMB = \angle BMP$, triangles DMB and BMP are also similar. It follows that $\angle CBD = \angle BPM = \angle ABK$. Now triangles BAH and BCA are also similar. Since $CD = DA$, we have $AK = KH$.



4. Let a copy F of the convex polygon be placed anywhere inside the square. Consider the copy F' obtained from F by a half-turn about the centre O of the square. By hypothesis, F and F' must have a point in common. Let it be P. Then the point P' obtained from P by a half-turn about O is also in the intersection of F and F'. Since F is convex, O is also in F. It follows that a copy of the convex polygon placed anywhere inside the square must cover O. It follows that if three copies are placed in the square, they will have O in common.

5. The columns may be permuted in 7! ways so that the first row is different. The remaining rows may be permuted in 6! ways so that the first column is different. Once the first row and the first column have been fixed, the remaining entries in the table are also fixed. Hence the total number of different tables we can generate is $7! \times 6!$.