International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2008.**

1. In the convex hexagon $ABCDEF$, AB, BC and CD are respectively parallel to DE, EF and FA. If $AB = DE$, prove that $BC = EF$ and $CD = FA$.

2. There are ten congruent segments on a plane. Each point of intersection divides every segment passing through it in the ratio 3:4. Find the maximum number of points of intersection.

3. There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

4. Find all positive integers n such that $(n+1)!$ is divisible by $1! + 2! + \cdots + n!$.

5. Each cell of a 10×10 board is painted red, blue or white, with exactly twenty of them red. No two adjacent cells are painted in the same colour. A domino consists of two adjacent cells, and it is said to be good if one cell is blue and the other is white.

   (a) Prove that it is always possible to cut out 30 good dominoes from such a board.

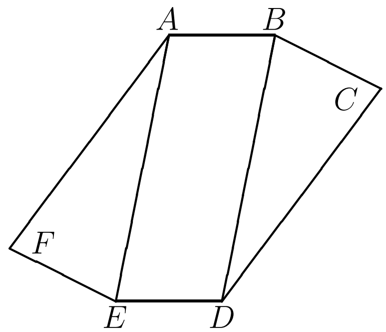
   (b) Give an example of such a board from which it is possible to cut out 40 good dominoes.

   (c) Give an example of such a board from which it is not possible to cut out more than 30 good dominoes.

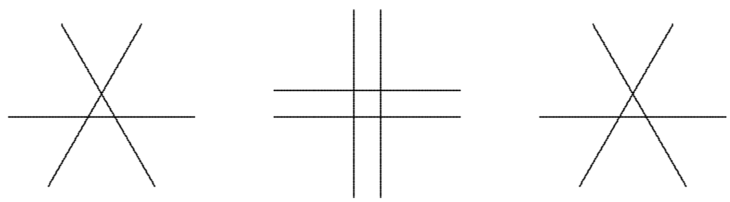
Note: The problems are worth 4, 5, 5, 5 and 6 points respectively.

Solution to Junior O-Level Spring 2008

1. Since AB and DE are equal and parallel, $ABDE$ is a parallelogram so that $AE = BD$. Moreover, AE is parallel to BD. Since CD is parallel to FA, $\angle CDB = \angle FAE$. Similarly, $\angle CBD = \angle FEA$. Hence triangles BCD and EFA are congruent, so that $BC = EF$ and $CD = FA$.



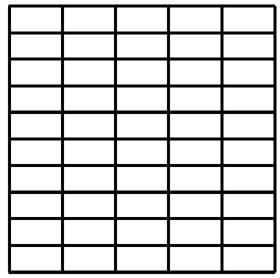
2. On each segment, there are exactly two points which divide it in the ratio 3:4. Hence the total count segment by segment is at most 20. However, it takes two segments to produce a point of intersection. Hence there are at most 10 such points. The diagram below shows how this can be attained, so that 10 is indeed the maximum.



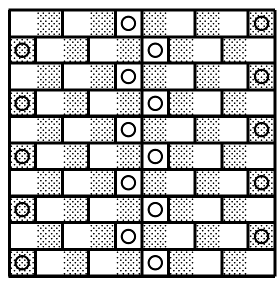
3. Suppose none of a, b and c is 0. They cannot all be positive and they cannot be all negative. By symmetry, we may assume that a and b are positive while c is negative. Since a and b are distinct, we may assume that $a > b$. If $a > -c$, we take five cards with a on each. Then it is impossible to take another five cards to bring the total down to 0. If $-c > a$, we take five cards with c on each. Then it is impossible to take another five cards to bring the total up to 0. It follows that we must have $a = -c > b$. If we now take four cards with a on each and a fifth card with b on it, it is impossible to take another five cards to bring the total down to 0.

4. For $n = 1$, $1!$ divides $2!$. For $n = 2$, $1!+2!$ divides $3!$. We claim that there are no solutions for $n \geq 3$. We have
$$\begin{aligned}(n+1)! &= n! + n(n!)\\ &= n((n-1)! + n!)\\ &< n(1! + 2! + \cdots + n!).\end{aligned}$$
For $n = 3$, $4! > 2(1! + 2! + 3!)$. Suppose for some $n \geq 3$, $n! > (n-2)(1! + 2! + \cdots + (n-1)!)$. Note that $2(n-2) \geq n-1$. By mathematical induction,
$$\begin{aligned}(n+1)! &= (n-1)n! + 2n!\\ &> (n-1)n! + 2(n-2)(1! + 2! + \cdots + (n-1)!)\\ &\geq (n-1)(1! + 2! + \cdots + n!).\end{aligned}$$
It follows that $\dfrac{(n+1)!}{1! + 2! + \cdots + n!}$ lies strictly between $n-1$ and n. Hence it cannot be an integer, and the claim is justified.

5. (a) Divide the board into 50 dominoes as shown in the diagram below. At most 20 of them can contain a red cell. Each of the other 30 must be a good domino since no two adjacent cells are painted in the same colour.



(b) Paint the board blue and white in the usual checkerboard pattern as shown in the diagram below, where the blue cells are shaded. Repaint into red cells 20 of the cells marked by circles, and divide the rest of the board into 40 dominoes, each of which is good.



(c) Divide the board into 50 dominoes as in (a) and paint the board as in (b). If we repaint any 20 blue cells into red cells, we will only have 30 blue cells left. Since we need a blue cell in each good domino, we will have exactly 30 good dominoes.