International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2008.**

1. An integer N is the product of two consecutive integers.
   (a) Prove that we can add two digits to the right of this number and obtain a perfect square.
   (b) Prove that this can be done in only one way if $N > 12$.

2. A line parallel to the side AC of triangle ABC cuts the side AB at K and the side BC at M. O is the point of intersection of AM and CK. If $AK = AO$ and $KM = MC$, prove that $AM = KB$.

3. Alice and Brian are playing a game on a $1 \times (N + 2)$ board. To start the game, Alice places a checker on any of the N interior squares. In each move, Brian chooses a positive integer n. Alice must move the checker to the n-th square on the left or the right of its current position. If the checker moves off the board, Alice wins. If it lands on either of the end squares, Brian wins. If it lands on another interior square, the game proceeds to the next move. For which values of N does Brian have a strategy which allows him to win the game in a finite number of moves?

4. Given are finitely many points in the plane, no three on a line. They are painted in four colours, with at least one point of each colour. Prove that there exist three triangles, distinct but not necessarily disjoint, such that the three vertices of each triangle have different colours, and none of them contains a coloured point in its interior.

5. Standing in a circle are 99 girls, each with a candy. In each move, each girl gives her candy to either neighbour. If a girl receives two candies in the same move, she eats one of them. What is the minimum number of moves after which only one candy remains?

6. Do there exist positive integers $a,\ b,\ c$ and d such that $\frac{a}{b} + \frac{c}{d} = 1$ and $\frac{a}{d} + \frac{c}{b} = 2008$?

7. A convex quadrilateral $ABCD$ has no parallel sides. The angles between the diagonal AC and the four sides are $55^\circ,\ 55^\circ, 19^\circ$ and 16° in some order. Determine all possible values of the acute angle between AC and BD.

Note: The problems are worth 2+2, 5, 6, 6, 7, 7 and 8 points respectively.