International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2008.**

1. A standard 8×8 chessboard is modified by varying the distances between parallel grid lines, so that the cells are rectangles which are not necessarily squares, and do not necessarily have constant area. The ratio between the area of any white cell and the area of any black cell is at most 2. Determine the maximum possible ratio of the total area of the white cells to the total area of the black cells.

2. Space is dissected into non-overlapping unit cubes. Is it necessarily true that for each of these cubes, there exists another one sharing a common face with it?

3. A two-player game has $n > 2$ piles each initially consisting of a single nut. The players take turns choosing two piles containing numbers of nuts relatively prime to each other, and merging the two piles into one. The player who cannot make a move loses the game. For each n, determine the player with a winning strategy, regardless of how the opponent may respond.

4. In the quadrilateral $ABCD$, AD is parallel to BC but $AB \neq CD$. The diagonal AC meets the circumcircle of triangle BCD again at A' and the circumcircle of triangle BAD again at C'. The diagonal BD meets the circumcircle of triangle ABC again at D' and the circumcircle of triangle ADC again at B'. Prove that the quadrilateral $A'B'C'D'$ also has a pair of parallel sides.

5. In the infinite sequence $\{a_n\}$, $a_0 = 0$. For $n \geq 1$, if the greatest odd divisor of n is congruent modulo 4 to 1, then $a_n = a_{n-1} + 1$, but if the greatest odd divisor of n is congruent modulo 4 to 3, then $a_n = a_{n-1} - 1$. The initial terms are 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2 and 1. Prove that every positive integer appears infinitely many times in this sequence.

6. $P(x)$ is a polynomial with real coefficients such that there exist infinitely many pairs (m, n) of integers satisfying $P(m)+P(n) = 0$. Prove that the graph $y = P(x)$ has a centre of symmetry.

7. A contest consists of 30 true or false questions. Victor knows nothing about the subject matter. He may write the contest several times, with exactly the same questions, and is told the number of questions he has answered correctly each time. How can he be sure that he will answer all 30 questions correctly

   (a) on his 30th attempt;

   (b) on his 25th attempt?

Note: The problems are worth 4, 6, 6, 6, 8, 9 and 5+5 points respectively.

Solution to Senior A-Level Fall 2008[1]

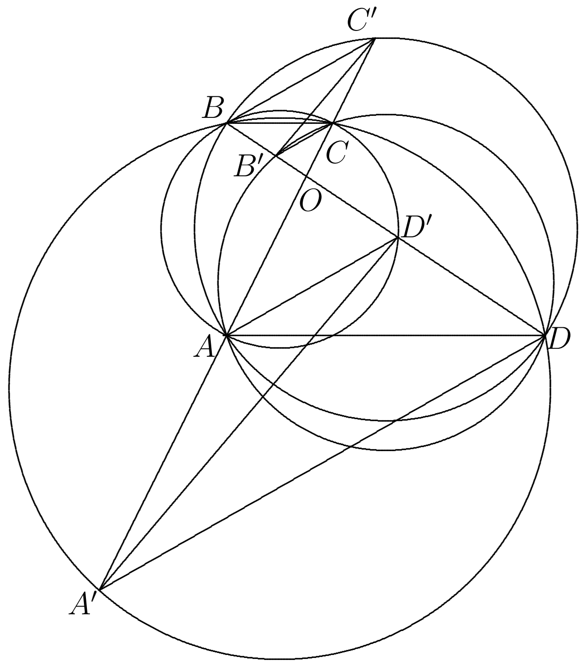
1. We may have rows and columns of alternating widths $\frac{1}{3}$ and $\frac{2}{3}$. Let the white cells remain squares while the black cells become non-squares. Then the area of each white cell is either $\frac{1}{9}$ or $\frac{4}{9}$ while the area of each black cell is $\frac{2}{9}$. Thus the ratio between the area of any white cell and the area of any black cell is at most 2. The total area of the white cells is $16(\frac{1}{9}+\frac{4}{9})=\frac{80}{9}$ while the total area of the black cells is $32(\frac{2}{9})=\frac{64}{9}$. Here, the ratio of the total area of the white cells to the total area of the black cells is $\frac{80}{9}:\frac{64}{9}=5:4$. To show that this is the maximum possible, divide the modified chessboard into 16 subboards each consisting of four cells in a 2×2 configuration. Let the dimensions of one of the subboards be $a\times b$. Let the vertical grid line divide it into two rectangles of widths x and $a-x$, and we may assume that $x>\frac{a}{2}$. Let the horizontal grid line divide the subboard into two rectangles of heights y and $b-y$, and we may assume that $y>\frac{b}{2}$. The condition that the ratio between the area of any white cell and the area of any black cell is at most 2 applies here also, and this is satisfied if and only if $x\leq\frac{2a}{3}$ and $y\leq\frac{2b}{3}$. Let the white cells be $x\times y$ and $(a-x)(b-y)$. Their total area is $T=2xy+ab-bx-ay=x(2y-b)+a(b-y)$. Since $2y>b$, T increases as x increases to its maximum value of $\frac{2a}{3}$. Similarly, $T=y(2x-a)+b(a-x)$ increases as y increases to its maximum value of $\frac{2b}{3}$. Hence $T\leq\frac{8ab}{9}+ab-2\frac{2ab}{3}=\frac{5ab}{9}$. It follows the ratio of the total area of the white cells to the total area of the black cells is at most 5:4. Since this is true in each of the 16 subboards, it is true on the entire board.

2. It is not necessarily true. We tile space in the standard way with unit cubes. Choose one of them and call it C. There are two cubes sharing the front and back faces of C. They belong to two infinite columns of cubes parallel to the x-axis. There are two cubes sharing the top and bottom faces of C. They belong to two infinite columns of cubes parallel to the y-axis. There are two cubes sharing the left and right faces of C. They belong to two infinite columns of cubes parallel to the z-axis. These six columns do not intersect. Now we shift all the cubes in each column by half a unit. Then C does not share a complete face with any other cube.

3. Call a pile *even* if it has an even number of nuts. If it has an odd number of nuts, call it *small* if it has only one, and *big* otherwise. We first consider the case when n is odd. The first player is forced to form an even pile with two nuts on the first move, and the second player merges it with a small pile into a big pile. The second player's strategy is to always leave one big pile and an even number of small piles for the first player. In each move, the first player is forced to form an even pile, and it will be the only even pile at the time. If this merger involves the big pile, the second player merges the even pile with a small pile. If this merger does not involve the big pile, then the even pile has two nuts, and 2 is relatively prime to any odd number, in particular, the number of nuts in the big pile. Thus the second player can merge the big pile with the even pile. It follows that the second player always has a move, and hence wins the game. We now consider the case where n is even. the second player uses exactly the same strategy until the game is down to one big pile and three small piles (or four small piles at the beginning of the game for $n=4$). After the first player creates once again an even pile, the second player merges the other two piles into a second even pile. Now the first player has no move, and the second player wins.

[1]Courtesy of Andy Liu.

4. We have $\angle CAD' = \angle CBD' = \angle CA'D$ and $\angle ACB' = \angle ADB' = \angle AC'B$ from the cyclic quadrilaterals. Since AD is parallel to BC, all six angles are equal. Hence BC', $B'C, AD'$ and $A'D$ are all parallel. Let O be the point of intersection of AC and BD. From similar triangles, we have $\frac{OB'}{OC} = \frac{OD'}{OA}$, $\frac{OC}{OB} = \frac{OA}{OD}$ and $\frac{OB}{OC'} = \frac{OD}{OA'}$. Multiplication yields $\frac{OB'}{OC'} = \frac{OD'}{OA'}$, so that triangles $OB'C'$ and $OD'A'$ are similar. It follows that $A'D'$ is parallel to $B'C'$.



5. **Solution by Mariya Sardarli.**

We first prove by induction on m that $a_n = 1$ whenever $n = 2^m - 1$. It is easy to verify that $a_1 = 1,\ a_2 = 2,\ a_3 = 1,\ a_4 = 2,\ a_5 = 3,\ a_6 = 2$ and $a_7 = 1$. Suppose that the result holds for some $m \geq 3$. For $1 \leq k \leq 2^m - 1, k \neq 2^{m-1}$, let $k = 2^s g$ where g is odd and $s \leq m - 2$. Then $2^m + k = 2^s(2^{m-s} + g)$ and $2^{m-s} + g \equiv g \pmod 4$ since $m - s \geq 2$. On the other hand, the greatest odd divisor of 2^{m-1} is 1 while that of $2^m + 2^{m-1} = 2^{m-1}3$ is 3. Hence $a_{2^{m+1}-1} - a_{2^m}$ is 2 less than $a_{2^m-1} - a_0$. By the induction hypothesis, $a_{2^m-1} = 1$. Since the greastest odd divisor of 2^m is 1, $a_{2^m} = 2$. It follows that $a_{2^{m+1}-1} = 2 + 1 - 0 - 2 = 1$. We claim that if a value h appears in the sequence, then the value $h + 2$ also appears in the sequence. Since 1 and 2 appear in the sequence, every positive integer appears in the sequence. Let $a_k = h$ for some k and let m be such that $k < 2^m$. As before, $a_k - a_0 = a_{2^{m+1}+k} - a_{2^{m+1}}$, so that $a_{2^{m+1}+k} = h + 2$. Thus the claim is justified and the sequence is unbounded. Suppose there is a value h which appears only a finite number of times. Every time the sequence hits a new high, it has to return to 1 at some point. After the last appearance of h, the sequence either cannot return to 1 or cannot get to a new high. This is a contradiction.

6. **Solution by Olga Ivrii.**
Let $P(x) = a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$. We may assume that $a_0 > 0$. Suppose m and n are of the same sign, say positive. Note that $P(x) > 0$ whenever $x > \alpha$ for some positive number α. In order for $P(m) + P(n) = 0$ to hold, we must have $P(m) \leq 0$ or $P(n) \leq 0$. By symmetry, we may assume the former is the case. Then $m < \alpha$, so that we have finitely many values of m. For each of these values of m, we have finitely many values of n for which $P(n) = -P(m)$. This contradicts the condition that there exist infinitely many pairs (m, n) of integers satisfying $P(m) + P(n) = 0$. It follows that m and n must have opposite signs. If k is even, then $P(x) > 0$ whenever $|x| > \alpha$ for some positive number α, and we have the same contradiction as before. It follows that k must be odd.

For odd k, $m^k + n^k = (m+n)Q_1(m, n)$, where $Q_1(m, n) = m^{k-1} - m^{k-2}n + \cdots - mn^{k-2} + n^{k-1}$. Note, that if $mn < 0$ then $Q_1(m, n) \geq m^{k-1} + n^{k-1}$.

Therefore, $0 = P(m) + P(n) = (m + n)Q_1(m, n) + Q_2(m, n)$ implies

$|m + n| = |Q_2(m, n)/Q_1(m, n)| < \beta$. ($Q_2(m, n)$ is polynomial of degree at most $k - 1$)

In case $mn > 0$, $P(m) + P(n) = 0$ implies that both $|m| < \beta$, $|n| < \beta$.

Since there exist infinitely many pairs (m, n) of integers satisfying $P(m) + P(n) = 0$, some value of $m+n$ must occur infinitely often. Let this value be c. Define $R(x) = P(x)+P(c-x)$. Then $R(x)$ has infinitely many roots. Since it is a polynomial, it is identically zero. Hence $P(x) + P(c - x) = 0$ for all real numbers x, meaning that the graph $y = P(x)$ has a centre of symmetry at $(\frac{c}{2}, 0)$.

7. (a) In the first test, Victor answers True for all 30 questions. Suppose he gets 15 questions correct. In the second test, Victor changes the answers in test 1 to Questions 2, 3 and 4. The number of correct answers must be 12, 18, 14 or 16. In the first two cases, Victor knows the correct answers to Questions 2, 3 and 4, and has enough tests left to sort out the remaining questions. Hence we may assume by symmetry that the number of correct answers is 14. This means that the correct answers to two of Questions 2, 3 and 4 are True, and the other one False. Victor then changes the answers from test 1 to Questions $2k - 1$ and $2k$ in the k-th test, $3 \leq k \leq 15$. If in the kth test, he gets either 13 or 17 questions correct, then he knows the correct answers to Questions $2k - 1$ and $2k$. Thus we may assume that he gets 15 correct answers in each test. Thus one correct answer is True and other False in each of these 13 pairs of questions. Moreover, Victor now knows that the correct answer to Question 1 is False. So far, he has used 15 tests. In test 16, Victor changes the answers from test 1 to Questions 2, 3 and 5, and in test 17 to Questions 2, 4 and 5. The following chart shows he can deduce the correct answers to Questions 2 to 6. He has just enough tests left to sort out the remaining pairs.

Correct Answer to Question					Numbers of Correct Answers in Test	
2	3	4	5	6	16	17
T	T	F	T	F	12	14
T	F	T	T	F	14	12
F	T	T	T	F	14	14
T	T	F	F	T	14	16
T	F	T	F	T	16	14
F	T	T	F	T	16	16

Suppose that in the first test, Victor gets a correct answers where $a \neq 15$. He changes the answers from test 1 to Questions $2k-1$ and $2k$ in the k-th test, $2 \leq k \leq 15$. In each of these 14 tests, he will get either a questions correct again, or $a \pm 2$ questions correct. In the latter case, he will know the correct answers to Questions $2k-1$ and $2k$. In the former case, he will know that one of these two answers is True and the other is False. Victor will also have similar knowledge about Questions 1 and 2 since he knows the total number of answers that should be True. Because $a \neq 15$, Victor must know the correct answers to one pair of questions. Hence he only needs at most 14 more questions to sort out the remaining pairs.

(b) **Solution by Nhan Nyugen.**
In the first test, Victor answers True for all 30 questions. Suppose he gets a questions correct. In the second test, he changes the answers to the first two questions to False. He will get either a questions correct again, or $a \pm 2$ questions correct. In the latter case, he will know the correct answers to the first two questions. In the former case, Victor changes the first four answers to True, False, False and False in the third test, and to False, True, False and True in the fourth test. In the third test, the number of correct answers may be $a \pm 1$ or $a \pm 3$, while in the fourth test, the number of correct answers may be a or $a \pm 2$. From these data, he can deduce the correct answers to the first four questions, as shown in the chart below.

Number of Correct Answers		Correct Answer to Question			
in the Third Test	in the Fourth Test	1	2	3	4
$a-3$	a	False	True	True	True
	$a-2$	False	True	False	True
$a-1$	a	False	True	True	False
	$a+2$	True	False	True	True
	$a-2$	True	False	True	False
$a+1$	a	True	False	False	True
	$a+2$	False	True	False	False
$a+3$	a	True	False	False	False

Victor now handles each of the six subsequent groups of four questions in the same manner in 3 more questions, because Question 1 is relevant throughout. After 22 tests, he knows all the correct answers except to the last two questions. He can use the 23rd test to determine the correct answer to the second last question. Then he also knows the answer to the last question because he knows the total number of answers that should be True. Thus in the 24th test, Victor can answer all 30 questions correctly.