International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2008.**

1. Arrange the boxes in a line so that the number of pencils in them increases from left to right. Then the first box from the left contains at least one pencil, the next one contains at least two pencils, ..., the tenth box from the left contains at least 10 pencils. Take any pencil from the first box. Since the second box contains pencils of at least two different colors, some of these pencils has color distinct from that of the chosen pencil. Take it. The third box contains pencils of at least three colors. Hence some of these pencils has color distinct from the colors of both chosen pencils. Take it. Proceeding in the same manner, we choose the required 10 pencils of different colors.

2. Subtract 50 from each given number exceeding 50. By the conditions of the problem, each of the resulting differences is distinct from any of 25 given numbers not exceeding 50. So these numbers and the differences form a set of 50 distinct positive integers not exceeding 50. Thus it contains all positive integers from 1 to 50. Their sum equals $51 \cdot 25$, hence the sum of the given numbers equals $51 \cdot 25 + 50 \cdot 25 = 101 \cdot 25 = 2525$.

3. Let B_1, B_2, B_3 be the midpoints of arcs A_1A_2, A_2A_3, A_3A_1, respectively. The area of hexagon $A_1B_1A_2B_2A_3B_3$ is the sum of the areas of quadrilaterals $OA_1B_1A_2$, $OA_2B_2A_3$, and $OA_3B_3A_1$. But each of these quadrilaterals has perpendicular diagonals, hence the area of each quadrilateral is the half-product of its diagonals. Therefore, the required sum is equal to $\frac{1}{2}OB_1 \cdot A_1A_2 + \frac{1}{2}OB_2 \cdot A_2A_3 + \frac{1}{2}OB_3 \cdot A_3A_1$. Since $OB_1 = OB_2 = OB_3 = 2$ by the conditions of the problem, this sum is numerically equal to $A_1A_2 + A_2A_3 + A_3A_1$, as required.

4. Answer. Yes, it can.

   Solution. First take any three distinct positive integers such that one of them is equal to the half-sum of the remaining two; for instance, 1, 2, and 3. Their product equals 6 and so is not 2008th power of a positive integer. Multiply each of these numbers by 6^n to obtain 6^n, $2 \cdot 6^n$, $3 \cdot 6^n$. As before, one of the numbers is the half-sum of two others, and now their product equals 6^{3n+1}. It remains to choose n so that $3n+1$ equals 2008 (or is divisible by 2008). Since 2007 is divisible by 3, we can take $3n + 1 = 2008$, that is, $n = 669$.

5. Represent the running track as the left half of a circle. We may assume that a runner at the end of the running track does not turn back but continues to run along the right half of the same circle. Thus all runners are always running along this circle. The condition that they are at the same point of the initial running track means that they are on a line perpendicular to the diameter separating the left and right halves of the circle. Suppose all runners meet (are on the corresponding line) in time t after start. Then all runners in the left and in the right halves are at the same distance x from the starting point. Each runner in the left half has covered some integer number of circles plus distance x, and each runner in the right half has to run distance x to cover some integer number of circles. Where will the runners be in time $2t$ after start? Each runner in the left half will cover some integer number of circles plus distance $2x$, and each runner in the right half will have to run distance $2x$ to cover some integer number of circles. But this means that they again will be on a line perpendicular to the diameter separating the left and right halves of the circle, because they are at the same distance (along the circle) from the starting point. Hence, on the initial running track, the runners will meet again in time $2t$, similarly in time $3t$, and so on.