International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2008.**

1. On a 100×100 chessboard, 100 Queens are placed so that no two attack each other. Prove that if the board is divided into four 50×50 subboards, then there is at least one Queen in each subboard.

2. Each of four stones weighs an integral number of grams. Available for use is a balance which shows the difference of the weights between the objects in the left pan and those in the right pan. Is it possible to determine the weight of each stone by using this balance four times, if it may make a mistake of 1 gram either way in at most one weighing?

3. Serge has drawn triangle ABC and one of its medians AD. When informed of the ratio $\frac{AD}{AC}$, Elias is able to prove that $\angle CAB$ is obtuse and $\angle BAD$ is acute. Determine the ratio $\frac{AD}{AC}$ and justify your result.

4. Baron Münchausen asserts that he has a map of Oz showing five towns and ten roads, each road connecting exactly two cities. A road may intersect at most one other road once. The four roads connected to each town are alternately red and yellow. Can this assertion be true?

5. Let $a_1,\ a_2,\ \ldots,\ a_n$ be positive numbers such that $a_1 + a_2 + \cdots + a_n \le \frac{1}{2}$. Prove that $(1 + a_1)(1 + a_2)\cdots(1 + a_n) < 2$.

6. ABC is a non-isosceles triangle. E and F are points outside triangle ABC such that $\angle ECA = \angle EAC = \angle FAB = \angle FBA = \theta$. The line through A perpendicular to EF intersects the perpendicular bisector of BC at D. Determine $\angle BDC$.

7. In the infinite sequence $\{a_n\}$, $a_0 = 0$. For $n \ge 1$, if the greatest odd divisor of n is congruent modulo 4 to 1, then $a_n = a_{n-1} + 1$, but if the greatest odd divisor of n is congruent modulo 4 to 3, then $a_n = a_{n-1} - 1$. The initial terms are 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2 and 1.

   (a) Prove that the number 1 appears infinitely many times in this sequence.

   (b) Prove that every positive integer appears infinitely many times in this sequence.

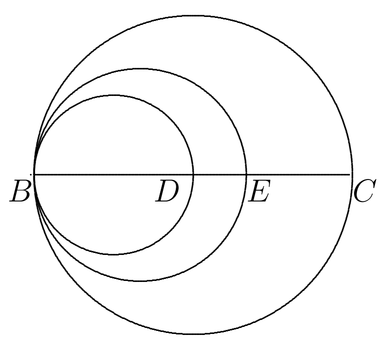
Note: The problems are worth 4, 6, 6, 6, 8, 9 and 5+5 points respectively.

Solution to Junior A-Level Fall 2008[1]

1. Clearly, the western half of the board has 50 Queens, as does the northern half of the board. Assume by symmetry that the northwestern quadrant is empty. Then 50 Queens must be in the southwestern quadrant and another 50 in the northeastern quadrant. Hence the southeastern quadrant is also empty. However, the squares in the southwestern and the northeastern quadrants all lie on 99 diagonals going between the southwest and the northeast. By the Pigeonhole Principle, two of the Queens will be on squares of the same diagonal, and hence attack each other. This is a contradiction. Hence no quadrants may be empty.

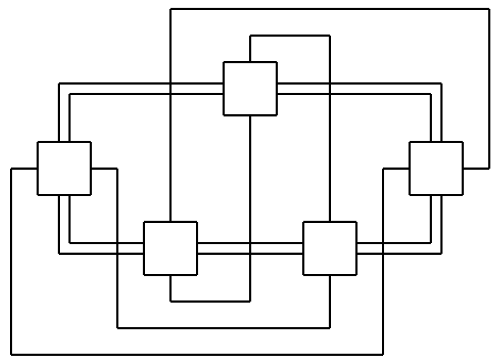
2. Label the coins A, B, C and D, with respective weights a, b, c and d. In the four weighings, weigh B, C and D, A and B against C and D, A and C against B and D, and A and D against B and C. Let the results be $b+c+d=w$, $a+b-c-d=x$, $a-b+c-d=y$ and $a-b-c+d=z$. For now, assume that no mistakes are possible. We have $w+x+y+z=3a$ so that $a=\frac{w+x+y+z}{3}$. Since $y+z=2a-2b$, we have $b=\frac{2a-(y+z)}{2}$. Similarly, $c=\frac{2a-(z+x)}{2}$ and $d=\frac{2a-(x+y)}{2}$. Suppose now a mistake of 1 gram is possible. If $w+x+y+z$ is a multiple of 3, no mistakes have been made. If it is one more or one less, we know the direction of the mistake. In any case, we can round the total to the nearest multiple of 3 and use it to determine a. Now each of $a-w$, x, y and z has the same parity as $a+b+c+d$, and hence as one another. Whichever has the opposite parity to the other three is where the mistake has been made.

3. Note that $\angle CAB$ is obtuse if and only if A lies inside the circle with diameter BC, and $\angle BAD$ is acute if and only if A lies outside the circle with diameter BD. If $\frac{AD}{AC}$ is constant, A lies on a circle of Apollonius of C and D, with a diameter on the line CD. The only such circle which lies between the circles with diameters BC and BD must also have B as one end of the diameter on CD. Since $\frac{BD}{BC}=\frac{1}{2}$, the other end of this diameter must be the point E between C and D such that $\frac{ED}{EC}=\frac{1}{2}$. It follows that the ratio Serge gives Elias must be $\frac{1}{2}$. Since ABC is a triangle, A does not lie on BC. Then $\angle CAB$ is obtuse since A lies inside the circle with diameter BC, and $\angle BAD$ is acute since A lies outside the circle with diameter BD.



[1]Courtesy of Andy Liu.

4. **Solution by Sasha Kitaygorodsky.**
The Baron may have the following map. The double lines indicate the yellow brick road while the single lines indicate the red side-streets.



5. **Solution by Mariya Sardarli.**
By the Arithmetic-Geometric Means Inequality, we have

$$(1+a_1)(1+a_2)\cdots(1+a_n) \le \left(\frac{(1+a_1)+(1+a_2)+\cdots+(1+a_n)}{n}\right)^n \le \left(1+\frac{1}{2n}\right)^n.$$

By the Binomial Theorem,

$$\begin{aligned}\left(1+\frac{1}{m}\right)^m &= \sum_{k=0}^{m}\binom{m}{k}\frac{1}{m^k}\\ &< \sum_{k=0}^{n}\frac{1}{m!}\\ &\le \frac{1}{0!}+\frac{1}{1!}+\sum_{k=2}^{m}\left(\frac{1}{k-1}-\frac{1}{k}\right)\\ &= 3-\frac{1}{m}.\end{aligned}$$

Hence $(1+\frac{1}{2n})^n = \sqrt{(1+\frac{1}{2n})^{2n}} < \sqrt{3} < 2$.

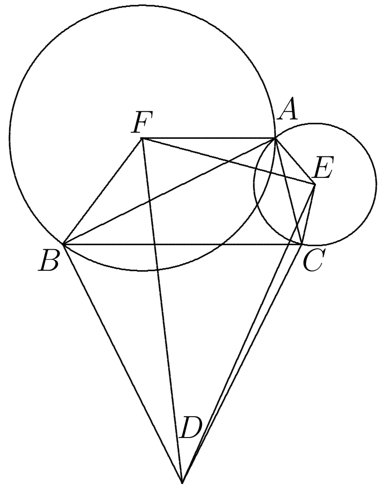
6. Let D be the point outside triangle ABC such that $DB = DC$ and $\angle BDC = 2\theta$. Draw the circles through A with respective centres E and F. The powers of the point D with respect to these circles are $DE^2 - CE^2$ and $DF^2 - BF^2$. We claim that they are equal, so that D lies on the common chord of these circles, which passes through A and is perpendicular to the line EF of centres. By the Cosine Law,

$$DE^2 - CE^2 = DC^2 - 2DC\cdot CE\cos DCE = DC^2 + 2DC\cdot CE\sin BCA$$

since $\angle DCB + \angle ACE = 90°$. Similarly,

$$DF^2 - BF^2 = DB^2 - 2DB\cdot BF\cos DBF = DB^2 + 2DB\cdot BF\sin ABC.$$

Since triangles ACE and ABF are similar, $\frac{CE}{BF} = \frac{AC}{AB}$. By the Sine Law, $\frac{AC}{AB} = \frac{\sin BCA}{\sin ABC}$. Hence $CE\sin BCA = BF\sin ABC$. Since $DC = DB$, we indeed have $DE^2 - CE^2 = DF^2 - BF^2$ and the claim is justified. Since the position of D is uniquely determined, $\angle BDC = 2\theta$.



7. **Solution by Mariya Sardarli.**

(a) We will prove by induction on m that $a_n = 1$ whenever $n = 2^m - 1$. It is easy to verify that $a_1 = 1$, $a_2 = 2$, $a_3 = 1$, $a_4 = 2$, $a_5 = 3$, $a_6 = 2$ and $a_7 = 1$. Suppose that the result holds for some $m \geq 3$. For $1 \leq k \leq 2^m - 1$, $k \neq 2^{m-1}$, let $k = 2^s g$ where g is odd and $s \leq m-2$. Then $2^m + k = 2^s(2^{m-s} + g)$ and $2^{m-s} + g \equiv g \pmod 4$ since $m - s \geq 2$. On the other hand, the greatest odd divisor of 2^{m-1} is 1 while that of $2^m + 2^{m-1} = 2^{m-1}3$ is 3. Hence $a_{2^{m+1}-1} - a_{2^m}$ is 2 less than $a_{2^m-1} - a_0$. By the induction hypothesis, $a_{2^m-1} = 1$. Since the greastest odd divisor of 2^m is 1, $a_{2^m} = 2$. It follows that $a_{2^{m+1}-1} = 2 + 1 - 0 - 2 = 1$.

(b) We claim that if a value h appears in the sequence, then the value $h+2$ also appears in the sequence. Since 1 and 2 appear in the sequence, every positive integer appears in the sequence. Let $a_k = h$ for some k and let m be such that $k < 2^m$. By the argument in (a), $a_k - a_0 = a_{2^{m+1}+k} - a_{2^{m+1}}$, so that $a_{2^{m+1}+k} = h + 2$. Thus the claim is justified and the sequence is unbounded. Suppose there is a value h which appears only a finite number of times. Every time the sequence hits a new high, it has to return to 1 at some point by the result in (a). After the last appearance of h, the sequence either cannot return to 1 or cannot get to a new high. This is a contradiction.