International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2008**

1 [**4**] 100 Queens are placed on a 100×100 chessboard so that no two attack each other. Prove that each of four 50×50 corners of the board contains at least one Queen.

2 [**6**] Each of 4 stones weights the integer number of grams. A balance with arrow indicates the difference of weights on the left and the right sides of it. Is it possible to determine the weights of all stones in 4 weighings, if the balance can make a mistake in 1 gram in at most one weighing?

3 [**6**] In his triangle ABC Serge made some measurements and informed Ilias about the lengths of median AD and side AC. Based on these data Ilias proved the assertion: angle CAB is obtuse, while angle DAB is acute. Determine a ratio AD/AC and prove Ilias' assertion (for any triangle with such a ratio).

4 [**6**] Baron Münchausen claims that he got a map of a country that consists of five cities. Each two cities are connected by a direct road. Each road intersects no more than one another road (and no more than once). On the map, the roads are colored in yellow or red, and while circling any city (along its border) one can notice that the colors of crossed roads alternate. Can Baron's claim be true?

5 [**8**] Let $a_1, \dots, a_n$ be a sequence of positive numbers, so that $a_1 + a_2 + \dots + a_n \leq 1/2$. Prove that
$(1 + a_1)(1 + a_2) \dots (1 + a_n) < 2.$

6 [**9**] Let ABC be a non-isosceles triangle. Two isosceles triangles $AB'C$ with base AC and $CA'B$ with base BC are constructed outside of triangle ABC. Both triangles have the same base angle φ. Let C_1 be a point of intersection of the perpendicular from C to $A'B'$ and the perpendicular bisector of the segment AB. Determine the value of $\angle AC_1B$.

7 In an infinite sequence $a_1, a_2, a_3, \dots$, the number a_1 equals 1, and each a_n, $n > 1$, is obtained from a_{n-1} as follows:

- if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,
- and if this residue equals 3, then $a_n = a_{n-1} - 1$.

Prove that in this sequence

(a) [**5**] the number 1 occurs infinitely many times;

(b) [**5**] each positive integer occurs infinitely many times.
(The initial terms of this sequence are 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, ...)