International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Spring 2007.**

1. A 9×9 chessboard with the standard checkered pattern has white squares at its four corners. What is the least number of rooks that can be placed on this board so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)

2. The polynomial $x^3 + px^2 + qx + r$ has three roots in the interval (0,2). Prove that $-2 < p + q + r < 0$.

3. B is a point on the line which is tangent to a circle at the point A. The line segment AB is rotated about the centre of the circle through some angle to the line segment $A'B'$. Prove that the line AA' passes through the midpoint of BB'.

4. A binary sequence is constructed as follows. If the sum of the digits of the positive integer k is even, the k-th term of the sequence is 0. Otherwise, it is 1. Prove that this sequence is not periodic.

5. A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can fit together in the box without turning either piece over. How can this be done if

   (a) one angle of the triangle is obtuse and is twice as big as one of the acute angles;

   (b) the angles of the triangle are $20°$, $30°$ and $130°$?

Note: The problems are worth 3, 4, 4, 4 and 3+3 points respectively.

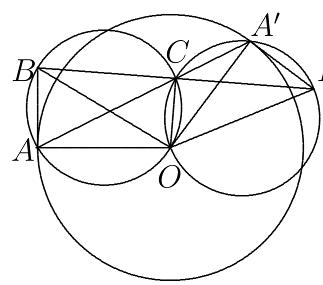
[1] Courtesy of Professor Andy Liu

Solution to Senior O-Level Spring 2007

1. A rook on a black square attacks exactly 9 white squares. A rook on a white square attacks either 7 or 9 white squares. Since there are 41 white squares overall, we need at least 5 rooks. Labelling the ranks a to i and the files 1 to 9, we can show that 5 is indeed the minimum by placing a rook on each of a8, c6, e4, g2 and i2.

2. Let the roots be a, b and c. Then $x^3+px^2+qx+r=(x-a)(x-b)(x-c)$. Putting in $x=1$, we have $1+p+q+r=(1-a)(1-b)(1-c)$. We are given that each of a, b and c lies strictly between 0 and 2. Hence each of $1-a$, $1-b$ and $1-c$ lies strictly between -1 and 1, and it follows that so does their product. Hence $p+q+r=(1-a)(1-b)(1-c)-1$ lies strictly btween -2 and 0.

3. Let O be the centre of the circle and C be the midpoint of BB'. Then OC is perpendicular to BB'. Since $\angle OAB=\angle OCB=90^\circ=\angle OCB'=\angle OA'B'$, both $OABC$ and $OCA'B'$ are cyclic quadrilaterals. Hence $\angle ACB=\angle AOB=\angle A'OB'=\angle A'CB'$. Since B, B' and C are collinear, so are A, A' and C. In other words, the line AA' passes through the midpoint C of BB'.

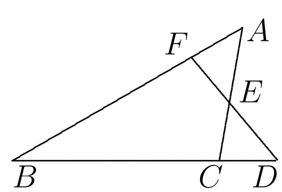


4. We define a sequence S as follows. If the sum of the digits of the non-negative integer k is even, the k-th term of S is 0. Otherwise, it is 1. It is the same sequence except for the inclusion of a 0-th term, and S is periodic if and only if so is the original sequence. Now each block of 10 consecutive terms of S is (0101010101) or (1010101010). We can contract the former to 0 and the latter to 1 and call the new sequence S'. Note that if the sum of the digits of the positive integer $10k$ is even, the k-th term of S' is 0. Otherwise, it is 1. Suppose S is periodic with period p. Because of the contraction, S' is periodic with period $\frac{p}{10}$. This is a contradiction since S' is identical to S. Hence S cannot be periodic.

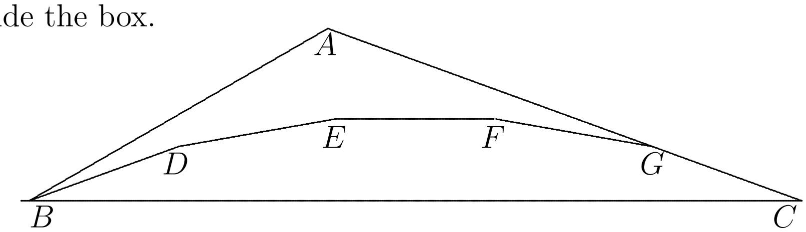
5. (a) Let the pie be represented by triangle ABC in which $\angle CAB=\theta$ and $\angle BCA=2\theta$. Let D and F be the respective images of A and C under the reflection about the bisector of $\angle ABC$. (See the diagram below to the left.) Let AC and DF intersect at E. Then triangle DBF represents the box. Now $\angle BDF=\angle CAB=\theta$. It follows that

$$\angle DEC=\angle BCA-\angle BDF=\theta=\angle BDF.$$

Hence triangle CDE is isosceles, as is triangle FAE which is congruent to it by reflection. It follows that if we cut the pie along EF, we can fit the two pieces inside the box.



(b) **First Solution:**
Let $BDEFGC$ be a hexagon with $\angle BDE = \angle DEF = \angle EFG = \angle FGC = 170°$, $\angle DBC = \angle BCG = 20°$ and $BD = DE = EF = FG = GC$. Extend GC to A so that $\angle ABC = 30°$. Then the hexagon $ABDEFG$, though non-convex, has a bilateral symmetry just as the convex hexagon $BDEFGC$ does. Thus it may be moved so that its four equal sides coincide with DE, EF, FG and GC respectively, forming with $BDEFGC$ a triangular box which is the mirror-image of the pie ABC. (See the diagram below.) If we cut the pie along the polygonal line $B - D - E - F - G$, we can fit the two pieces inside the box.



Second Solution by Olga Ivrii:
Let $ACDEF$ be a pentagon with $\angle CDE = \angle DEF = \angle EFA = 130°$, $AC = DC$ and $DE = EF = FA$. Extend AF and CD to meet at B. Then the quadrilateral $BDEF$, though non-convex, has a bilateral symmetry just as the convex pentagon $ACDEF$ does. Thus it may be moved so that its two short sides coincide with EF and FA respectively, forming with $ACDEF$ a triangular box which is the mirror-image of the pie ABC. (See the diagram below.) If we cut the pie along the polygonal line $D - E - F$, we can fit the two pieces inside the box.