International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2007.**

1. A, B, C and D are points on the parabola $y = x^2$ such that AB and CD intersect on the y-axis. Determine the x-coordinate of D in terms of the x-coordinates of A, B and C, which are a, b and c respectively.

2. A convex figure F is such that any equilateral triangle with side 1 has a parallel translation that takes all its vertices to the boundary of F. Is F necessarily a circle?

3. Let $f(x)$ be a polynomial of nonzero degree. Can it happen that for any real number a, an even number of real numbers satisfy the equation $f(x) = a$?

4. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop."

   (a) Can Andy guarantee that after he says "stop," no card is in its initial spot?

   (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

5. From a regular octahedron with edge 1, cut off a pyramid about each vertex. The base of each pyramid is a square with edge $\frac{1}{3}$. Can copies of the polyhedron so obtained, whose faces are either regular hexagons or squares, be used to tile space?

6. Let a_0 be an irrational number such that $0 < a_0 < \frac{1}{2}$. Define $a_n = \min\{2a_{n-1}, 1 - 2a_{n-1}\}$ for $n \geq 1$.

   (a) Prove that $a_n < \frac{3}{16}$ for some n.

   (b) Can it happen that $a_n > \frac{7}{40}$ for all n?

7. T is a point on the plane of triangle ABC such that $\angle ATB = \angle BTC = \angle CTA = 120°$. Prove that the lines symmetric to AT, BT and CT with respect to BC, CA and AB, respectively, are concurrent.

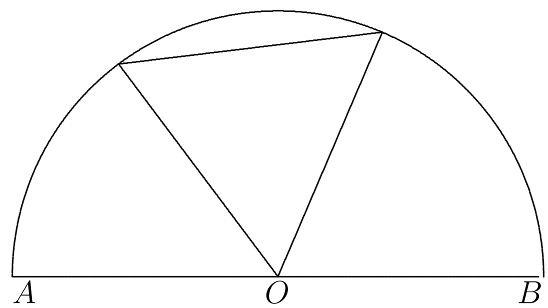
Note: The problems are worth 3, 5, 5, 4+4, 8, 4+4 and 8 points respectively.

[1] Courtesy of Professor Andy Liu

Solution to Senior A-Level Spring 2007

1. Let $(t,0)$ be the point of intersection of AB and CD. Then the equation of the line AB is given by $\frac{y}{x-t}=\frac{a^2-b^2}{a-b}=a+b$. That A lies on this line means that $\frac{a^2}{a-t}=a+b$. We have $a^2=a^2+ab-t(a+b)$ so that $t=\frac{ab}{a+b}$. Similarly, $t=\frac{cd}{c+d}$. Eliminating t, we have $abc+abd=d(ac+bc)$ so that $d=\frac{abc}{ac+bc-ab}$.

2. **Solution by Olga Ivrii.**
No, the convex figure does not have to be a circle. Let AB be a horizontal segment of length 2. Draw a semicircle with diameter AB above AB. For any equilateral triangle of side 1, place its lowest vertex at the midpoint O of AB. If there are two choices, place either one at O. The other two vertices of the equilateral triangle always lie on the semicircle. Hence the convex figure bounded by AB and the semicircle has the desired property.

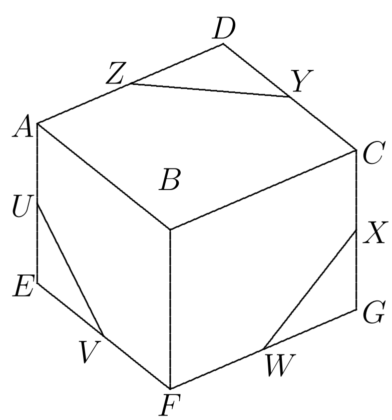


3. The graph of $f(x)$ is continuous and may have a number of turning points, that is, points at which the graph changes from increasing to decreasing or vice versa. Let these points be $(x_i, f(x_i))$ for $1 \leq i \leq n$. By symmetry, we may assume that the leading coefficient of $f(x)$ is positive. Suppose the degree of $f(x)$ is odd. Then $f(x)$ tends to ∞ as x tends to ∞ and to $-\infty$ as x tends to $-\infty$. Let a be a real number such that $a > f(x_i)$ for $1 \leq i \leq n$. Then the equation $f(x) = a$ is satisfied by exactly one real number x. Suppose the degree of $f(x)$ is even. Then $f(x)$ tends to ∞ as x tends to $\pm\infty$. It follows that n is odd, so that there is a real number a for which $a = f(x_i)$ for an odd number of i, $1 \leq i \leq n$. Then the equation $f(x) = a$ is satisfied by an even number of real numbers x where $x \neq x_i$ for $1 \leq i \leq n$, as well as by an odd number of x_i. Hence the equation is satisfied by an odd number of real numbers.

4. (a) The answer is yes. Andy can call the cards out in order starting with the Ace of Spades, two of Spades down to the King of Spades, followed by the Hearts, the Diamonds and the Clubs. We refer to this as one cycle. In each cycle, each card can move at most once since it is called exactly once, and at least one card must move. Andy then makes another 51 cycles of calls. We claim that all moves are in the same direction, either all clockwise or all counter-clockwise. This is clear within each cycle. Consider the card X which is the last to move in a cycle, and let Y be the other card adjacent to the empty spot. Since Y does not move after X in this cycle, it must have been called before X. So in the next cycle, Y will be called before X, and follows X in the same direction. This justifies our claim. To go once around and return to its initial spot, a card must have moved 53 times, and this is not possible since Andy makes only 52 cycles of calls. If it is to be in its initial spot, it must not have moved at all. However, this is also impossible as otherwise at most 1 move could have been made, but in 52 cycles, at least 52 moves have been made. Therefore, after 52 cycles of calls, every card is in a spot different from its initial one.

(b) The answer is no. Construct a graph where each of the vertices represents one of the 52! permutations of the cards, with the first and the last adjacent to the empty spot. Two vertices are joined by an edge if and only if a call by Andy changes the two permutations to each other. Label the edge with the card called by Andy. In this graph, each vertex has degree 2, and the graph is a union of disjoint cycles. Consider the cycle containing the vertex representing the initial permutation. For each vertex, let a person starts there. Whenever Andy makes a call, the person moves along an edge labelled with that card to an adjacent vertex if possible, and stays put otherwise. We call a vertex safe if and only if in the permutation it represents, the Queen of Spades is not adjacent to the empty spot. By shifting each card clockwise into the empty spot in turns, we will arrive at permutations represented by safe vertices as well as permutations represented by unsafe vertices. Note that after each call, there is still one person on each vertex. Thus no matter what sequence of calls Andy may employ, he cannot get everyone to a safe vertex. It follows that there is an initial permutation for which Andy's sequence will leave the Queen of Spades adjacent to the empty spot.

5. The answer is yes. Let $ABCD-EFGH$ be a cube, with U, V, W, X, Y and Z the respective midpoints of AE, EF, FG, GC, CD and DA, as shown in the diagram below. Now UV, VW and YZ are all parallel to AC. Since UX, VY and WZ are concurrent at the centre of the cube, $UVWXYZ$ is a planar hexagon, and a regular one by symmetry. This planar section cuts the cube into two congruent halves. Each half contains a vertex where three mutually perpendicular faces meet. We call it the primary vertex. (In the diagram, one of them is B and the other one is the hidden vertex H.) If we glue eight copies of this half-cube together so that their primary vertices coincide, we obtain a copy of the solid in question, the one with six square and eight regular hexagonal faces. Divide space into cubes by the planes $x = k$, $y = k$ and $z = k$, where k runs through all integers. Dissect each cube into halves so that the primary vertices have either all odd co-ordinates or all even co-ordinates. If we glue eight copies of the half-cube around each primary vertex, we have a partition of space into copies of the solid in question.



6. **Official Solution.**

(a) We consider five cases.
Case 1. $0 < a_0 < \frac{3}{16}$.
Here, we already have $a_0 < \frac{3}{16}$.
Case 2. $\frac{3}{16} < a_0 < \frac{1}{5}$.
Let $a_0 = \frac{1}{5} - \epsilon$ where $0 < \epsilon < \frac{1}{80}$. Then $a_1 = \frac{2}{5} - 2\epsilon$, $a_2 = \frac{1}{5} + 4\epsilon$, $a_3 = \frac{2}{5} + 8\epsilon$ and $a_4 = \frac{1}{5} - 16\epsilon$. Suppose $\frac{3}{16} < a_{4k}$ for all k. then $a_{4k} = \frac{1}{5} - 16^k\epsilon$. This is a contradiction since ϵ is a fixed positive number.

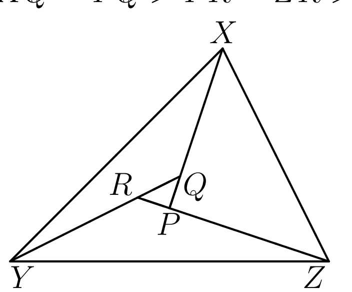
Case 3. $\frac{1}{5} < a_0 < \frac{1}{4}$.
Let $a_0 = \frac{1}{4} - \epsilon$ where $0 < \epsilon < \frac{1}{20}$. Then $a_1 = \frac{1}{2} - 2\epsilon$ and $a_2 = 4\epsilon < \frac{1}{5}$. Hence either Case 1 or Case 2 applies with a_2 playing the role of a_0.
Case 4. $\frac{1}{4} < a_0 < \frac{1}{3}$.
Let $a_0 = \frac{1}{3} - \epsilon$ where $0 < \epsilon < \frac{1}{12}$. Then $a_1 = \frac{1}{3} + 2\epsilon$ and $a_2 = \frac{1}{3} - 4\epsilon$. Suppose $\frac{1}{4} < a_{2k}$ for all k. then $a_{2k} = \frac{1}{4} - 4^k\epsilon$. This is a contradiction since ϵ is a fixed positive number.
Case 5. $\frac{1}{3} < a_0 < \frac{1}{2}$.
Here, we have $a_1 < \frac{1}{3}$. Hence one of Case 1, Case 2, Case 3 and Case 4 applies, with a_1 playing the role of a_0.

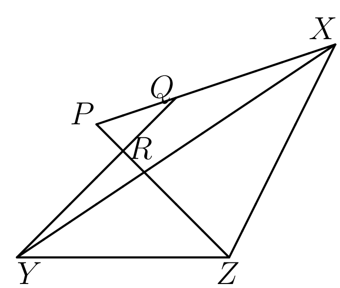
(b) It is possible. Call a number ϵ good if $\frac{\epsilon}{3}$ is obtained from $\frac{1}{2^7} + \frac{1}{2^{15}} + \frac{1}{2^{23}} + \cdots + \frac{1}{2^{8k+7}} + \cdots$ by omitting some terms in a non-periodic manner. Since the sum of the infinite geometric series is $\frac{1}{2^7}\left(\frac{1}{1-\frac{1}{2^8}}\right) = \frac{2}{255}$, ϵ is an irrational number satisfying $0 < \epsilon < \frac{2}{85}$. Now let $a_0 = \frac{1}{5} - \delta$ for some good number δ. We consider two cases.
Case 1. The term $\frac{1}{2^7}$ is absent from $\frac{\delta}{3}$.
Let $\epsilon = 2^8\delta$. Then ϵ is also a good number. We have $a_0 = \frac{1}{5} - \frac{\epsilon}{2^8}$, $a_1 = \frac{2}{5} - \frac{\epsilon}{2^7}$, $a_2 = \frac{1}{5} + \frac{\epsilon}{2^6}$, $a_3 = \frac{2}{5} + \frac{\epsilon}{2^5}$, $a_4 = \frac{1}{5} - \frac{\epsilon}{2^4}$, $a_5 = \frac{2}{5} - \frac{\epsilon}{2^3}$, $a_6 = \frac{1}{5} + \frac{\epsilon}{2^2}$, $a_7 = \frac{1}{5} + \frac{\epsilon}{2}$ and $a_8 = \frac{1}{5} - \epsilon$.
Case 2. The term $\frac{1}{2^7}$ is present in $\frac{\delta}{3}$.
Let $\epsilon = 2^8(\delta - \frac{3}{2^7})$. Then ϵ is also a good number. We have $a_0 = \frac{113}{640} - \frac{\epsilon}{2^8}$, $a_1 = \frac{113}{320} - \frac{\epsilon}{2^7}$, $a_2 = \frac{47}{160} + \frac{\epsilon}{2^6}$, $a_3 = \frac{33}{80} - \frac{\epsilon}{2^5}$, $a_4 = \frac{7}{40} + \frac{\epsilon}{2^4}$, $a_5 = \frac{7}{20} + \frac{\epsilon}{2^3}$, $a_6 = \frac{3}{10} - \frac{\epsilon}{2^2}$, $a_7 = \frac{2}{5} + \frac{\epsilon}{2}$ and $a_8 = \frac{1}{5} - \epsilon$.
Note that in both cases, $a_n > \frac{7}{40}$ for $0 \le n \le 7$. Moreover, a_8 has the same form as a_0, so that either Case 1 or Case 2 applies with a_8 playing the role of a_0. It follows that $a_n > \frac{7}{40}$ for all n.

7. **Solution by Yan Li and Andy Liu.**
We first establish a preliminary result.

Lemma.
XP, YQ and ZR are pairwise intersecting rays such that $\angle QYZ = \angle RZY$, $\angle RZX = \angle PXZ$ and $\angle PXY = \angle QYX$. Then XP, YQ and ZR are concurrent.

Proof:
Suppose that, to the contrary, they intersect at three points. Consider first the case where all three rays are directed towards the opposite sides of the triangle, as shown in the diagram below on the left. Then QYZ, PZX and RXY are isosceles triangles. We have a contradiction since $XP > XQ = YQ > YR = ZR > ZP = XP$.





Consider now the case where only one ray is directed towards the opposite side of the triangle, as shown in the diagram above on the right. Then RYZ, PZX and QXY are isosceles triangles. We have a contradiction since

$$QR = YQ - YR = XQ - ZR = (XP - PQ) - (ZP - PR) = PR - PQ.$$

We now tackle the given problem. Since the rays TA, TB and TC make angles of $120°$ with one another, the point T is inside the triangle. Let the extensions of AT, BT and CT intersect the opposite sides at P, Q and R respectively. Let X, Y and Z be the images of T under reflections across BC, CA and AB respectively. Then $AZ = AT = AY$. Hence $\angle AYZ = \angle AZY$. Also, $\angle AYQ = \angle ATQ = 60° = \angle ATR = \angle AZR$. Hence

$$\angle QYZ = \angle QYA - \angle ZYA = \angle RZA - \angle YZA = \angle RZY.$$

Similarly, $\angle RZX = \angle PXZ$ and $\angle PXY = \angle QYX$. By the Lemma, XP, YQ and ZR are concurrent.