International Mathematics

TOURNAMENT OF THE TOWNS

Senior P-Level Paper **Fall 2007.**

1 [**1**] A straight line is colored with two colors. Prove that there are three points A, B, C of the same color such that $AB = BC$.

SOLUTION. Consider any two points of the same color; say white, W and W'; let $2d$ be the distance between them and W' be to the right of W. Consider two new points, on the distance $2d$ to the left of W and $2d$ to the of right of W'. Both of them must be black; otherwise, the problem is solved. Now, consider a midpoint between W and W'. It must be black as well; otherwise, the problem is solved. However, in this case this midpoint is equidistant from both black points. The statement is proven.

2 [**2**] A student did not notice multiplication sign between two three-digit numbers and wrote it as a six-digit number. Result was 7 times more that it should be. Find these numbers.

SOLUTION. Problem is equivalent to find two 3-digit numbers u, v, so that $1000u+v = 7u \times v$. Therefore, $u = v/(7v - 1000)$. Since $100 \geq u \geq 999$ then $100 \geq v/(7v - 1000) \geq 999$. Solving the last inequality we get $v = 143$. Then we find corresponding $u = 143$.

3 [**3**] Two players in turns color the squares of a 4×4 grid, one square at the time. Player loses if after his move a square of 2×2 is colored completely. Which of the players has the winning strategy, First or Second?

SOLUTION. Second Player has a strategy. On each move of First Player, Second Player corresponding move is two squares down (or two squares up) in the same column. It is easy to see that if First Player has a move, so does Second Player.

4 [**3**] There three piles of pebbles, containing 5, 49, and 51 pebbles respectively. It is allowed to combine any two piles into a new one or to split any pile consisting of even number of pebbles into two equal piles. Is it possible to have 105 piles with one pebble in each in the end?

ANSWER: it is not possible.

SOLUTION. It is clear that the first operation can be one of the following:

a). Combining 5 and 49;

b). Combining 5 and 51;

c). Combining 49 and 51.

Let us consider case a). After the first opearation is applied we have two piles: 54 and and 51. Note, that both piles are multiple of 3. If a number is multiple of 3 then dividing it by 2 (coprime with 3) results in a number that is multiple of 3. Adding two numbers multiple of 3 results in a number that is multiple of 3. Therefore, no matter which operation we apply from now on we can get only piles that all are multiple of 3. But 1 is not a multiple of 3. Therefore, in case a) it is impossible to get piles with one pebble in each.

Cases b) and c) are dealt in similar way (piles in case b are multiple of 7 while in case c are multiple of 5).

5 [**4**] Jim and Jane divide a triangular cake between themselves. Jim choses any point in the cake and Jane makes a straight cut through this point and choses the piece. Find the size of the piece that each of them can guarantee for himself/herself (both of them want to get as much as possible).

ANSWER. Jim can guarantee for himself 4/9 of the cake while Jane can guarantee 5/9.

SOLUTION. Jim chooses a point of intersections of medians of the triangle (centroid). It is easy to prove that if Jane makes parallel cut to a side (any side) then she get exactly 5/9 of the cake. If the cut is not parallel then she gets less (to the trapezoid we had in previous case one triangle is added while the other is subtracted; compare the areas of these triangles). Jim can not get more by choosing any other point. If it was the case, Jane can always make a cut parallel to one of the sides (so to choose a trapezoid with centroid in it) and have more than 5/9.