International Mathematics

TOURNAMENT OF THE TOWNS

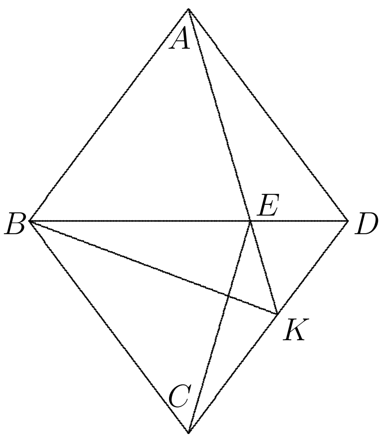
Junior A-Level Paper **Fall 2007.**

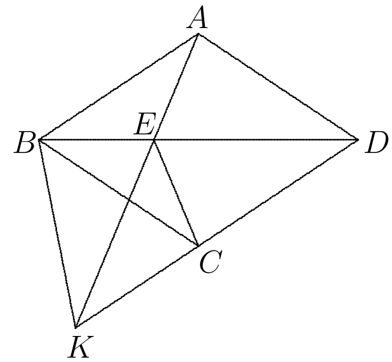
1. Let $ABCD$ be a rhombus. Let K be a point on the line CD, other than C or D, such that $AD = BK$. Let P be the point of intersection of BD with the perpendicular bisector of BC. Prove that A, K and P are collinear.
2. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same three numbers, prove that these three numbers are equal to each other.
   (b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead?
3. Michael is at the centre of a circle of radius 100 metres. Each minute, he will announce the direction in which he will be moving. Catherine can leave it as is, or change it to the opposite direction. Then Michael moves exactly 1 metre in the direction determined by Catherine. Does Michael have a strategy which guarantees that he can get out of the circle, even though Catherine will try to stop him?
4. Two players take turns entering a symbol in an empty cell of a $1 \times n$ chessboard, where n is an integer greater than 1. Aaron always enters the symbol X and Betty always enters the symbol O. Two identical symbols may not occupy adjacent cells. A player without a move loses the game. If Aaron goes first, which player has a winning strategy?
5. Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and have been replaced on the objects at random. We wish to determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible?
6. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.
   (a) Prove that if this is possible for some n, then it is also possible for $2n$.
   (b) Determine all n for which this is possible.
7. For each letter in the English alphabet, William assigns an English word which contains that letter. His first document consists only of the word assigned to the letter A. In each subsequent document, he replaces each letter of the preceding document by its assigned word. The fortieth document begins with "Till whatsoever star that guides my moving." Prove that this sentence reappears later in this document.

Note: The problems are worth 5, 3+3, 6, 7, 8, 4+5 and 9 points respectively.

Solution to Junior A-Level Fall 2007

1. Let AK intersect BD at E. We shall prove that $BE = CE$, so that E lies on the perpendicular bisector of BC. It will then follow that $E = P$, and that A, P and K are indeed collinear. Since $AD = BK$ and AB is parallel to DK, $ABKD$ is a cyclic quadrilateral. It follows that $\angle AKD = \angle ABD = \angle CBD$, so that $BCKE$ is also a cyclic quadrilateral. We now have $\angle ECB = \angle EKB = \angle ADB = \angle EBC$, so that $EB = EC$.





2. (a) **First Solution by Jarno Sun:**
   Let the numbers Peter thinks of be p_1, p_2 and p_3, the numbers Basil thinks of be b_1, b_2 and b_3, and the numbers both write down be w_1, w_2 and w_3. Note that each of $\gcd(w_1, w_2)$, $\gcd(w_2, w_3)$ and $\gcd(w_3, w_1)$ is equal to $\gcd(p_1, p_2, p_3)$. Similarly, each of lcm (w_1, w_2), $\text{lcm}(w_2, w_3)$ and $\text{lcm}(w_3, w_1)$ is equal to $\text{lcm}(b_1, b_2, b_3)$. It follows that $w_1w_2 = \gcd(w_1, w_2)\text{lcm}(w_1, w_2) = \gcd(w_2, w_3)\ \text{lcm}(w_2, w_3) = w_2w_3$, so that $w_1 = w_3$. Similarly, w_2 shares this common value.
   Second Solution:
   Let the numbers Peter thinks of be x, y and z. We assume to the contrary that $\gcd(x, y)$, $\gcd(x, z)$ and $\gcd(y, z)$ are not the same number. Then there must be a prime p such that the highest powers of p which divide these three numbers are not identical. Let the highest powers of p which divide x, y and z be a, b and c respectively. We may assume that $a \le b \le c$. Then the highest powers of p which divide $\gcd(x, y)$, $\gcd(x, z)$ and $\gcd(y, z)$ will respectively be a, a and b, and we have $a < b$. Now the highest power of p which divides any of Basil's numbers must be b, and p^b will divide two of his least common multiples. It follows that the two sets of three numbers cannot be identical unless all three numbers in each set are the same.

   (b) The answer is no. Peter's numbers may be 1, 2, 2 and 2. Then his six greatest common divisors are 1, 1, 1, 2, 2 and 2. Basil's numbers may be 1, 1, 1 and 2. Then his six least common multiples are 1, 1, 1, 2, 2 and 2. The two sets of numbers are identical, but the six numbers in each set are not all the same.

3. Michael can escape. In the first move, he chooses any direction. Catherine cannot gain anything by reversing it. In each subsequent move, Michael chooses a direction which is perpendicular to the line joining his current position to the centre of circle. Again Catherine cannot gain anything by reversing it. Let d_n be the distance of Michael from the centre of the circle after the n-th move. We have $d_1 = 1$ and $d_{n+1} = \sqrt{d_n^2 + 1}$. We claim that $d_n = \sqrt{n}$ for all $n \ge 1$. The basis holds, and by the induction hypothesis, $d_{n+1} = \sqrt{\sqrt{n}^2 + 1} = \sqrt{n+1}$. It follows that after 10000 moves, Michael will arrive at the circumference of the circle.

4. We claim that Betty can guarantee a win. We first prove the following auxiliary result. Suppose the first cell is marked X and the last cell is marked O, with n vacant cells in between. If Aaron goes first, he loses. We use induction on n. When $n = 1$, neither player has a move. Since Aaron moves first, he loses. Assume that result holds up to $n-1$ for some $n \geq 2$. Consider a board with n vacant cells between the X and O already marked. In his first move, Aaron will partition the board into two, with i and j vacant cells respectively, where $i + j = n - 1$. In the first board, the first and the last cells are marked X. Betty places an O next to either X. Then we have two boards each of which has X at one end and O at the other, with less than n vacant cells in between. Aaron will lose because by the induction hypothesis, he loses on both boards. We now return to the vacant $1 \times n$ board. Suppose Aaron marks an X on the k-th cell. By symmetry, we may assume that $k > 1$. Betty marks an O on the first cell. It is Aaron's move, and by the auxiliary result, he will lose if $k = n-1$ or n. If not, he will at some point be forced to mark an X on the ℓ-th cell where $\ell \geq k+2$. Then Betty will mark an O on the $(k+1)$st cell, and the auxiliary result applies again. Thus Aaron is forced to open up new parts of the board again and again. Eventually he runs out of room and loses. It follows that Betty can always win if $n \geq 2$.

5. **Solution by Dmitri Dziabenko.**
Let there be n objects, and let the mass indicated by the tag on the i-th object be m_i, $1 \leq i \leq n$. We may assume that $m_1 \leq m_2 \leq \cdots \leq m_n$. Arbitrarily choose positive numbers $d_2 < d_3 < \cdots < d_n$ and choose d_1 so that $d_1 m_1 = d_2 m_2 + d_3 m_3 + \cdots + d_n m_n$. On one side of the support, hang the 1-st object at a distance d_1 from the support. On the other side of the support, hang the i-th object at a distance d_i from the support for $2 \leq i \leq n$. Let the correct mass of the i-th object be a_i for $1 \leq i \leq n$. We consider three cases.
Case 1. All the tags are in fact correct.
Then we will have equilibrium.
Case 2. The tag on the 1-st object is correct but those on some of the others are not.
Then $\{a_2, a_3, \ldots, a_n\}$ is a permutation of $\{m_2, m_3, \ldots, m_n\}$, and by the Rearrangement Inequality,
$$d_2 a_2 + d_3 a_3 + \cdots + d_n a_n < d_2 m_2 + d_3 m_3 + \cdots + d_n m_n = d_1 m_1.$$
Case 3. The tag on the 1-st object is incorrect.
Then $m_1 < a_1 = m_j$ for some j, $2 \leq j \leq n$. Hence $d_1 a_1 = d_1 m_j > d_1 m_1$ while
$$\begin{aligned} & d_2 m_2 + d_3 m_3 + \cdots + d_n m_n \\ > \; & d_2 m_1 + d_3 m_2 + \cdots + d_j m_{j-1} + d_{j+1} m_{j+1} + \cdots + d_n m_n \\ \geq \; & d_2 a_2 + d_3 a_3 + \cdots + d_n a_n, \end{aligned}$$
and again we have no equilibrium.

6. (a) Given a row of n coins arbitrarily arranged heads and tails, and a number between 1 and n inclusive, the assistant can flip exactly one coin so that the magician can tell which number has been chosen. With a row of $2n$ coins and a number m between 1 and $2n$, the magician and the assistant place the numbers 1 to n in order in the first row of a $2 \times n$ array, and the numbers from $n+1$ to $2n$ in order in the second row. If the row number h and the column number k of the location of m are determined, then $m = (h-1)n + k$. The magician and the assistant also consider the $2n$ coins as in a $2 \times n$ array. Code each coin with heads up as 0 and each coin with tails up as 1. Compute the sum of the codes of the two coins in each column modulo 2 and regard the result as a linear array of n coded coins. By the hypothesis, the assistant can flip the q-th coded coin to signal the number k to the magician. This can be achieved by flipping either of the two coins in the q-th column. To signal the number h to the magician, the assistant will just use the bottom coin of the q-th column, code 0 meaning $h = 1$ and code 1 meaning $h = 2$. If the bottom coin is not correct, flip it. Otherwise, flip the top coin.

   (b) For $n = 1$, the assistant must flip the only coin. However, the chosen number can only be 1, and the magician does not require any assistance. Hence the task is possible. For $n = 2$, let the coins be coded as in (a). The assistant will just use the second coin, code 0 meaning $h = 1$ and code 1 meaning $h = 2$. If the second coin is not correct, flip it. Otherwise, flip the first coin. Hence the task is also possible. By (a), the task is possible whenever n is a power of 2. We now show that the converse also holds. Each of the 2^n arrangement of the coins codes a specific number between 1 and n. If n is not a power of 2, then $2^n = qn + r$ where q and r are the quotient and the remainder obtained from the Division Algorithm, with $r > 0$. By the Pigeonhole Principle, some number is coded by at most q arrangements. Each may be obtained by the flip of a single coin from exactly n other arrangements. The yields a total count of $qn < 2^n$. On the other hand, from each of the 2^n arrangements, we must be able to obtain one of these q arrangements by the flip of a single coin. This contradiction shows that the task is impossible unless n is a power of 2.

7. **Solution by Central Jury.**
   Let the i-th document be D_i, $0 \le i \le 40$. Note that D_0 consists only of the letter A, and D_1 consists only of the word assigned to the letter A. This word does not start with A, as otherwise all documents start with A, which is not the case since D_{40} starts with T. However, the word assigned to A does contain the letter A, so that D_1 contains D_0, and not from the beginning. Similarly, D_{i+1} contains D_i for $0 \le i \le 39$, again not from the beginning. Note that no D_i can start with a single-letter word, as otherwise this word will start all subsequent documents, which is not the case since D_{40} does not start with a single-letter word. Since there are only twenty-six letters in the English alphabet, D_j and D_k must start with the same letter for some j and k where $j < k \le 27$. Then D_{j+1} and D_{k+1} start with the same word, D_{j+2} and D_{k+2} start with the same two words, and so on. Let $t = 40 - k$. Since $k \le 27$, $t \ge 13$ so that D_{j+t} and $D_{k+t} = D_{40}$ start with the same thirteen words, including "Till whatsoever star that guides my moving". Since D_{40} contains a copy of D_{j+t} and not from the beginning, this sentence will reappear in D_{40}.