International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2006.**

1. Consider a convex polyhedron with 100 edges. All its vertices were cut off near themselves using sharp knives planes (it was done in such a way that these planes have no intersections inside or on the boundary of the polyhedron). Find out for the resulting polyhedron:

   (a) number of vertices,

   (b) number of edges.

   (G.A.Galperin)

   Answer. a) 200; b) 300.

   Solution. Observe that there are two vertices of the new polyhedron on each edge of the given one, and there are 3 edges starting in each vertex of the new polyhedron. Consequently, there are $2 \cdot 100 = 200$ vertices and $\frac{100 \cdot 3}{2} = 300$ edges in the resulting polyhedron.

2. Is it possible to find two such functions $p(x)$ and $q(x)$ that $p(x)$ is an even function, while $p\big(q(x)\big)$ is an odd function (other than identically equal to 0) ? (A.D. Blinkov, V.M. Gurovic)

   Answer. Yes, it is possible.

   Solution. Consider functions $p(x) = \cos x$ and $q(x) = \frac{\pi}{2} - x$. It is evident that $p(x)$ is an even function, while $p(q(x)) = \sin x$ is odd an odd function. There are also lots of different solutions.

3. Consider an arbitrary number $a > 0$. We know that the inequality $10 < a^x < 100$ has exactly 5 positive integer solutions. How many solutions in positive integers may have the inequality $100 < a^x < 1000$?

   Find all possibilities. (A.K. Tolpygo)

   Answer. 4,5 or 6.

   Solution. The inequality $10 < a^x < 100$ can be rewritten as $10 < 10^{bx} < 100$ or $1 < bx < 2$. Similarly $100 < ax < 1000$ is equivalent to $2 < bx < 3$. If n is the minimal integer solution of $1 < bx < 2$, then $b(n-1) < 1 < bn$ and $b(n+4) < 2 < b(n+5)$. Summing up the first inequality with itself and with the second one we obtain $b(2n-2) < 2 < b(2n)$ and $b(2n+3) < 3 < b(2n+5)$. Hence the inequality $2 < bx < 3$ has from 4 and up to 6 integer solutions ($2n, \dots, 2n+3$ are always solutions, while $2n-1$ and $2n+4$ may be and may not). Actually, all 3 cases are possible:

   - $b = \frac{5}{23}$; solutions of the first inequality are 5, 6, 7, 8, 9, solutions of the second one are 10, 11, 12, 13.
   - $b = \frac{5}{26}$; solutions of the first inequality are 6, 7, 8, 9, 10, solutions of the second one are 11, 12, 13, 14, 15.
   - $b = \frac{5}{27}$; solutions of the first inequality are 6, 7, 8, 9, 10, solutions of the second one are 11, 12, 13, 14, 15, 16.

4. Quadrangle $ABCD$ is inscribed and $AB = AD$. A point M lays on the side BC, while a point N lays on the side CD. Angle MAN equals to the half of the angle BAD. Prove that $MN = BM + ND$. (M.I.Malkin) SOLUTION 1. Denote by R the point symmetrical to B with respect to AM. Observe that in the same time R is symmetrical to D with respect to AN (since $AD = AB$ and angle MAN is equal to the sum of angles NAD and MAB). As $ABCD$ is inscribed, the sum of the angles ABC and ADC is equal to 180°. Consequently the sum of the angles ARM and ARN is equal to 180°, hence MRN is a straight line. Thus $BM + ND = MR + NR = MN$.

   SOLUTION 2. Rotate triangle MAB around the point A in such a way that AB coincides with AD. Denote by M' the image of the point M when we perform this rotation. Since $ABCD$ is inscribed the sum of the angles ABC and ADC is equal to 180° , so and the sum of the angles ADN and ADM' is equal to 180°. This means that $M'DN$ is a straight line. Triangles NAM' and NAM are equal by the equality of two sides and angle between them (angles NAM and $M'AN$ are equal since angle MAN is equal to the sum of NAD and MAB, AN is the common side, AM and AM' are equal by the construction). Consequently $MN = M'N = ND + DM' = ND + BM$.

5. Peter has n3 white $1 \times 1 \times 1$-cubes. He wants to make an $n \times n \times$-cube using them, and he wants to make this cube totally white from the outside. What is the minimum number of sides of the cubes Vasya has to paint in black to prevent Peter from doing this?

   (a) $n = 3$,
   (b) $n = 1000$

   (R.G.Zhenodarov)

   ANSWER. (a) 12; (b)1999999986.

   REMARK: In the $n \times n \times$-cube 8 corner bricks have 3 outside facets, $12(n-2)$ adjoining to the edges bricks have 2 outside facets, $6(n-2)^2$ bricks have one outside facets, other bricks have no outside facets. In order to prevent Peter from putting a cube into a corner, Vasya has to paint at least two its sides (opposite ones). In order to make it impossible to put a cube on the edge one has to paint at least 4 its facets (all but 2 opposite ones). In order to prevent Peter from putting a cube on the side Vasya has to paint all 6 sides of the cube.

   (a). SOLUTION. In the case $n = 3$ it is enough for Vasya to fully paint 2 cubes (12 sides) to prevent Peter from fulfilling his task as one of these sides will necessarily remain on the outside. If Vasya has painted not more than 11 sides, then Peter is able to choose 8 cubes that have not more than one painted side (otherwise the number of painted sides is not less than $2 \times (27 - 7) = 40$), then choose 12 cubes that have not more than 3 painted sides ($4\cdot(27-8-11) = 32 > 11$), and 6 cubes with not more than 5 painted sides ($6\cdot(27-8-12-5) = 12 > 11$). After that Peter will be able to put these cubes in the corners, edges and centers of the sides correspondingly and accomplish his task. A little different solution can be found in the 0-junior level, 8-9 grades, problem 5b.

   (b) SOLUTION. In the case $n = 1000$ it is enough for Vasya to paint two opposite sides of $1000^3 - 7$ cubes, i.e. 1999999986 sides. Then outside facet of one of the corner cubes will be painted in any construction of the big cube. In the same time, if Vasya has painted not more than $2 \cdot (1000^3 - 7) - 1 = 2 \cdot 1000^3 - 15$ sides, than Peter is able to choose 8 cubes with not more than 1 painted side ($2 \cdot (1000^3 - 7) > 2 \cdot 1000^3 - 15$), then choose $12 \cdot 998$, with not more than 3 painted sides ($4 \cdot (1000^3 - 8 - 12 \cdot 998 + 1) > 2 \cdot 1000^3 - 15$), and $6 \cdot 998^2$ with not more than 5 painted sides ($6 \cdot (1000^3 - 8 - 12 \cdot 998 - 6 \cdot 998^2 + 1) > 2 \cdot 1000^3 - 15$). Then Peter is able to construct white from the outside cube.