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Senior A-Level Paper **Spring 2006.**

1. Assume a convex polygon with 100 vertices is given. Prove that one can chose 50 points inside the polygon in such a way that every vertex lies on a line passing through two of the chosen points. (B.R.Frenkin)

   SOLUTION. Enumerate vertices of the polygon in a clockwise order: 1, ..., 100. Consider polygon consisting of 10 vertices: 1, 2, 21, 22, 41, 42, 61, 62, 81, 82. Its vertices lay on the 5 straight lines 1-22, 21-42, 41-62, 61-82, 81-2, which are given by 5 points of intersections (the first straight line with the second one, the second one with the third one ... the fifth one with the second one, it is evident, that all these points are different). Repeat this for the decagons with numbers of vertices that can be obtained from the numbers of considered decagon by adding 2, 4, ..., 18. This problem has lots of different solutions.

2. Do there exist positive integers n and k such that the decimal representation of 2^n contains the decimal representation of 5^k as its leftmost part, while the decimal representation of 5^n contains the decimal representation of 2^k as its leftmost part? (G.A.Galperin)

   ANSWER: No, they doesnt exist.

   SOLUTION If for some positive integer n the number 2^n starts by 5^k and the number 5^n by 2^k then this means that $5^k \times 10^s < 2^n < (5^k + 1) \times 10^s$ and $2^k \times 10^l < 5^n < (2^k + 1) \times 10^l$, thus $10^{k+l+s} < 10^n < 10^{k+l+s+1}$, which is impossible. (Last inequality $10^n < 10^{k+l+s+1}$ is true, because $5^k + 1 < 2 \times 5^k$ and $2^k + 1 < 5 \times 2^k$).

3. Consider the polynomial $P(x) = x^4 + x^3 - 3x^2 + x + 2$. Prove that for every positive integer k, the polynomial $P(x)^k$ has at least one negative coefficient. (M.I.Malkin)

   SOLUTION 1. Observe that for any polynomial $P(x)$ its value in the point $x = 1$ is equal to the sum of all coefficients. Consequently, the sum of the coefficients of the polynomial $P(x)^n$ is equal to $P(1)^n = (1+1-3+1+2)^n = 2^n$. But the free term of $P(x)^n$ is equal to $P(0)^n = 2^n$, while the coefficient at x^{4n} is equal to 1, and their sum is already $2^n + 1$. Hence one of the remaining coefficients of $P(x)^n$ is negative.

   SOLUTION 2. The coefficient at x^3 for the polynomial $P(x)^n$ can be obtained by adding n items $2^{n-1}x^3$ and $n(n-1)$ items $-3x^2 \times x \times 2^{n-2}$, consequently this coefficient is equal

   $$n \cdot 2^{n-1} - 3n(n-1)2^{n-2} = 2^{n-2}(-3n^2 + 5n) = n \cdot 2^{n-2}(-3n + 5),$$

   which is negative number for an arbitrary integer $n \geq 2$.

   SOLUTION 3. Observe that $P(0)^n = P(1)^n = 2^n$. But any polynomial F with positive coefficients is strongly monotonic when $x > 0$ (i.e. $x > y > 0 \implies F(x) > F(y) > 0$). This means that polynomial $P(x)^n$ has at least one negative coefficient.

4. Consider a triangle ABC, take the angle bisector AA', and assume given a point X on the interval AA'. Assume that the line BX intersects the line AC in a point denoted B', while the line CX intersects the line AB in a point denoted C'. Assume also that the intervals $A'B'$ and CC' meet in a point denoted P, and the intervals $A'C'$ and BB' meet in a point denoted Q. Prove that the angles PAC and QAB are equal. (M.A. Volchkevich)

Solution. Denote by $h_M(l)$ the distance from the point M to the straight line l. We will use the following simple

Lemma 1. *if three rays OL, OM and ON, are given then for all points K on the ray OM the ration $h_K(OL)/h_K(ON)$ is the same.*

For the solution of the given problem it is enough to prove that

$$h_P(AC)/h_P(AA') = h_Q(AB)/h_Q(AA')$$

(this equality together with equality of angles $A'AC$ and $A'BC$ means that angles PAC and QAB are equal). Using lemma we obtain $h_P(BC)/h_P(AC) = h_X(BC)/h_X(AC)$ and $h_Q(BC)/h_Q(AB) = h_X(BC)/h_X(AB)$, consequently (since $h_X(AC) = h_X(AB)$, because X lays on the bisector AA') $h_P(BC)/h_P(AC) = h_Q(BC)/h_Q(AB)$. So it is enough to prove that $h_P(BC)/h_P(AA') = h_Q(BC)/h_Q(AA')$. By lemma 1 the latter is equivalent to $h_{B'}(BC)/h_{B'}(AA') = h_{C'}(BC)/h_{C'}(AA')$.

Denote $\angle BAC = 2\alpha$. Observe that $h_{B'}(AB)/h_{B'}(AA') = \sin 2\alpha/\sin\alpha = h_{C'}(AC)/hC'(AA')$. Now it is enough to prove that $h_{B'}(BC)/h_{B'}(AB) = h_{C'}(BC)/h_{C'}(AC)$. Applying lemma again this transforms into $h_X(BC)/h_X(AB) = h_X(BC)/h_X(AC)$, which is evident (since $h_X(AC) = h_X(AB)$). The proof is finished.

5. Prove that there exist infinitely many pairs of integers with the following property: in the decimal representation of each integer, each digit is greater or equal to 7, and the product of the two integers in the pair is also an integer whose decimal representation has no digits less than 7. (S.I.Tokarev)

   Solution 1. All the pairs $(9\ldots 98877,\ 8\ldots 87)$ where in the first and second numbers amounts of the digits are equal are right for this problem. Their product (it can be shown using multiplication "in column") is equal to $8\ldots 878887\ldots 79899$ (there are $n-3$ eights at the beginning, then 7888, and then $n-3$ sevens).

   Solution 2. Consider numbers 877...7 (k-1 sevens) and 899...9987 (k-3 nines), their product is equal to the 7899...998788...8899 ($k-4$ nines and $k-2$ eights).

6. Twelve grasshoppers sit on a circle in 12 pairwise distinct points. These points split the circle into 12 arcs. When a signal is given, the grasshoppers jump simultaneously; each one jumps clockwise, from the endpoint of his arc to its midpoint. Thus 12 news are formed; then the signal is repeated, and so on. Is it possible that at least one grasshopper returns to his original position after he does

   (a) 12 jumps?

   (b) 13 jumps?

   (A.K.Tolpygo)

   Answer: (a),(b) No, it is not.

   (a) Solution 1. Let us call 12 simultaneous jumps of grasshoppers "turn". Assume that one of the grasshoppers (call him first) returned to the starting point (denote it by A) after 12 turns. Observe that order of the grasshoppers on the circle doesnt change. Thus the remaining 11 grasshoppers have jump over the point A (at least once) before the first grasshopper returns there. But in one turn not more than one grasshopper jumps over the point A, while in the first turn no grasshoppers jump over the point A! Consequently in 12 turns no more than 11 grasshoppers can jump over the point A, and the first one is not able to come back.

(a) SOLUTION 2. Observe that our situation is equivalent to the following one: we arrange the infinite amount of grasshoppers along the ray OM at the beginning placing 12 grasshoppers, just unrolling the circle into a segment by cutting it at the starting point of the grasshopper #1 (assume that clockwise bypass of the circle coincides with positive direction of the axis Ox). Then we think that the first grasshopper starts only at the left end of the segment (point 0). And attach to the right end the same segment with grasshoppers at the same points and so on (we obtained the ray with marked points $A_1, A_2, \dots$). In this new model grasshoppers jump in positive direction into the midpoint of the segment, connecting this and next grasshopper. Now we want to prove that after 12 jumps the first grasshopper is to the left from the point A_{13}.

Let us prove using induction that after n jumps the i-th grasshopper is at the centre of mass of the system

$$\big((A_i, C(0,n)g), (A_{i+1}, C(1,n)g), \dots, (A_{i+n}, C(n,n)g)\big)$$

(the first factor is the position of object, second is its mass, $C(k,n) = n!/(k! \cdot (n-k)!)$).

It is obvious that after first jump this proposition is true. Assume that after n jumps the i-th grasshopper is at the centre of mass of the system

$$\big((A_i, C(0,n)g), (A_{i+1}, C(1,n)g), \dots, (A_{i+n}, C(n,n)g)\big)$$

and the $(i+1)$-th at the centre of mass of

$$\big((A_{i+1}, C(0,n)g), (A_{i+2}, C(1,n)g), \dots, (A_{i+n+1}, C(n,n)g)\big).$$

Then the midpoint of the segment connecting them has the same coordinates as the centre of mass of the system

$$\Big(\text{C. of M.}\big((A_i, C(0,n)g), (A_{i+1}, C(1,n)g), \dots, (A_{i+n}, C(n,n)g)\big),$$
$$\text{C. of M.}\big((A_{i+1}, C(0,n)g), (A_{i+2}, C(1,n)g), \dots, (A_{i+n+1}, C(n,n)g)\big)\Big)$$

which is the same the centre of mass of

$$\big((A_i, C(0,n)g), (A_{i+1}, (C(1,n) + C(0,n))g), \dots,$$
$$(A_{i+n}, (C(n,n) + C(n-1,n))g)), (A_{i+n+1}, C(n,n)g)\big),$$

and this is the centre of mass of the system $(A_i, C(0,n+1)g), \dots (A_i + n + 1, C(n+1,n+1)g)$. Proposition is proved.

The proved proposition means that after 12 jumps the first grasshopper is in the centre of mass of the system $((A_1, C(0,12)g), \dots, (A_{13}, C(12,12)g)$. It is obvious that this point is inside the segment $[A_1, A_{13}]$.

(b). In this case after 13 jumps the first grasshopper is in the centre if mass of the system $((A_1, C(0,13)g), \dots, (A_{14}, C(13,13)g)$. But the same point can be represented as the centre of mass of two points with some masses in them: the first one is

$$\text{C. of M. } ((A_2, C(1,13)g), \dots, (A_{13}, C(12,13)g),$$

and the second one is

$$\text{C. of M. } ((A_1, C(0,13)g), (A_{14}, C(13,13)g).$$

It is evident that the first point is inside the segment $[A_1, A_13]$. Also $C(0,13) = C(13,13)$ and $A_1A_2 = A_{13}A_{14}$, hence the second point is inside the segment $[A_1, A_{13}]$ too. Consequently and the centre of mass of these two points with arbitrary masses is inside the segment $[A_1, A_{13}]$.

7. An ant crawls along a fixed closed trajectory along the edges of a dodecahedron, never turning back. The trajectory contains each edge of the dodecahedron exactly twice. Prove that the ant passes at least one edge in the same direction both times. (Reminder: a dodecahedron is a polyhedron with 20 vertices, 30 edges and 12 equal pentagonal faces; 3 faces meet at each vertex.) (A.V. Shapovalov)

Solution. Assume the trajectory passing through each edge in both directions exist. Consider a vertex A and three its neighbors B, C, D. Assume that at some moment of time the ant comes to the point A from the point B then after it he crawls to the point C or D. If he chose C, then at some other moment he comes from C to A and turns to D (otherwise there is $D \to A \to D$ in the trajectory, which is impossible). Similarly, when the ant comes from D to A he turns to B. Summing up, we proved that there are 2 kinds of crossroads ($B \to A \to C$, $C \to A \to D$, $D \to A \to B$) and (if at the beginning the ant choose not C but D) ($B \to A \to D$, $D \to A \to C$, $C \to A \to B$). This two kinds of crossroads can be described using the simple rule: in the first case ant always turns left at the crossroad, and in the second one he always turns right. Now mark out for each crossroad its type. Observe that if the ant starts his movement from some vertex going along some edge, then all its trajectory can be reconstructed using only these marks. So each collection of the marks on the vertices corresponds to some collection of closed and non-intersecting (by an edge, passing in the same direction) trajectories (although, we do not clam that this collection of trajectories is unique, we do not need this).

We assume that at the beginning there is one such closed trajectory passing through each edges two times. Now by turns change the marks on the vertices with the rule "turn to the left" to the marks "turn to the right". It is possible that after the very first operation our big trajectory splat into multiple. But it is evident that some closed trajectories that we obtain are unambiguously defined. Let us study how the amount of the trajectories can change when we change the marks. We want to prove that it remains odd. Suppose we have a crossroad ($B \to A \to C$, $C \to A \to D$, $D \to A \to B$). Consider different cases of the trajectories passing through A configurations:

(a) We have 3 different closed trajectories ($B \to A \to C \to \cdots \to B$), ($C \to A \to D \to \cdots \to C$), and ($D \to A \to B \to \cdots \to D$), then after the mark is changed we obtain ($C \to A \to B \to \cdots \to D \to A \to C \to \cdots \to B \to A \to D \to \cdots \to C$), so in this case total amount decreased by two and remained odd.

(b) We have 2 closed trajectories ($B \to A \to C \to \dots B$) and ($C \to A \to D \to \cdots \to D \to A \to B \to \cdots \to C$), after the mark change we obtain ($C \to A \to B \to \cdots \to C$) and ($B \to A \to D \to \cdots \to D \to A \to C \to \cdots \to B$), total amount does not change.

(c) We have one closed trajectory ($B \to A \to C \to \cdots \to C \to A \to D \to \cdots \to D \to A \to B \to \cdots \to B$) then after change we obtain ($B \to A \to D \to \cdots \to D \to A \to C \to \cdots \to C \to A \to B \to \cdots \to B$). Total amount does not change again.

Three more cases can be obtained by reversing the considered ones. And all other cases are just the same after the replacement of the notation. We proved that total amount of trajectories remains odd. But when all crossroads have marks “turn to the left” on them, the only way to divide the dodecahedron into closed trajectories is to go round each facet along its boundary (i.e. each trajectory consists of 5 edges and goes round one facet). It is evident that in this case we have 12 trajectories. We obtained the contradiction with oddity of their amount.