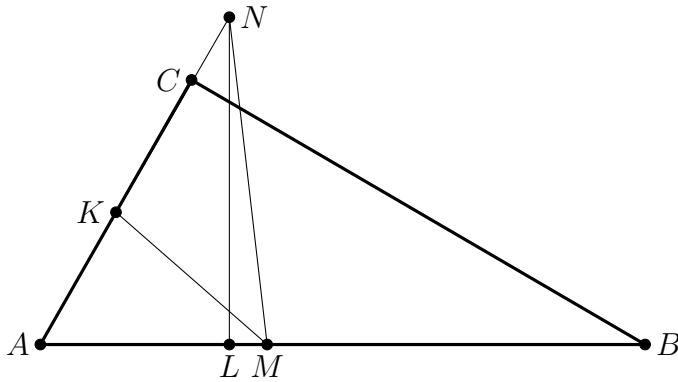## International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper   **Spring 2006.**

1. In triangle ABC angle A is equal to $60°$. The perpendicular from the midpoint of side AB intersects AC at the point N. The perpendicular from the midpoint of side AC intersects AB at the point M. Prove that $CB = MN$.   (R.G. Zhenodarov)

SOLUTION. By the property of the perpendicular from the midpoint $NA = NB$, thus triangle ANB is isosceles. Angle A is equal to $60°$, this means that triangle ANB is equilateral and $AN = AB$. Similarly, triangle AMC is equilateral, $AM = AC$. Triangles ACB and AMN are equal according to the equality of two sides and angle between them. Hence $BC = MN$.

2. Consider an $n \times n$ table. In each square of its first column someone has written the number 1, in each square of the second column, number 2, and so on. Then someone erased the numbers on the diagonal which connects top-left with bottom-right angle of the table. Prove that the sum of the numbers above the diagonal is twice the sum of the numbers under it. (S.A.Zaitsev)

SOLUTION 1. For each square on the diagonal compare the sums of the numbers situated to the left of it and situated above it. If the square is situated at the intersection of the k-th row and the k-th column the sum to the left is equal to $1 + 2 + \cdots + (k-1) = k(k-1)/2$, while the sum of the numbers above it is equal to k(k-1), that is two times more. Hence the sum of all numbers above the diagonal is two times more than the sum of the numbers situated to the left of it.

SOLUTION 2.

	2	3	4	...	n
1		3	4	...	n
1	2		4	...	n
1	2	3		...	n
...	...	...	...		...
1	2	3	4	...	

	1	2	3	...	$n-1$
1		1	2	...	$n-2$
1	2		1	...	$n-3$
1	2	3		...	$n-4$
...	...	...	...		...
1	2	3	4	...	

SOLUTION 3. In the original table (left picture) there are $(n-1)$ ones, $(n-2)$ twos, $(n-3)$ threes and so on. Let us subtract from each number above the diagonal the number symmetrical to it with respect to the diagonal. We get the picture to the right. It has equal numbers situated on the diagonals above the main one and parallel to it: $(n-1)$ ones, $(n-2)$ twos, $(n-3)$ threes and so on. We decreased the upper sum by the lower sum and got the

lower sum. This means that the upper sum was two times more than the lower one. We are going to show that after the subtraction of the lower sum from the upper sum we obtain the upper sum. From i-th row of the upper sum we subtract i-th column of the lower one. Since i-th row of the upper sum is given by $(i+1), (i+2), \ldots, (n-1), n$, while i-th column of the lower sum is $i, i, \ldots, i$, after the subtraction we get the row $1, 2, \ldots, (n-i)$, which is exactly the $(n-i+1)$-th line of the lower sum. After making the subtraction for every i we will obtain the lower sum. Consequently, the upper sum is two times more than the lower sum.

3. Consider an arbitrary number $a > 0$ such that the inequality $1 < xa < 2$ has exactly 3 integer solutions. How many integer solutions may have the inequality $2 < xa < 3$?

Find all possibilities. (A.K. Tolpygo)

ANSWER. 2, 3 or 4 solutions.

SOLUTION 1. The first inequality is equivalent to $1/a < x < 2/a$. There are 3 integers in the interval $(1/a, 2/a)$. They divide it into two segments 1 unit long each and two segments not more than 1 unit long each one along the edges. Therefore, the inequality $2 < 1/a \leq 4$ for the length of the segment $(1/a, 2/a)$ holds. Similarly if there are k integers in the segment t units long, then $k - 1 < t \leq k + 1$. It is equivalent to the inequality $t - 1 \leq k < t + 1$. Second segment $(2/a, 3/a)$ has the same length. Substituting $t = 1/a$ and taking into consideration the inequalities for $1/a$, we obtain $1 < 1/a - 1 \leq k < 1/a + 1 \leq 5$, i.e. $k = 2, 3$ or 4. All three cases are possible: $k = 2$ when $a = 3/8$ ($x = 6, 7$); $k = 3$ when $a = 1/4$ ($x = 9, 10, 11$); $k = 4$ when $a = 5/17$ ($x = 7, 8, 9, 10$).

SOLUTION 2. This problem also can be solved using graphical methods. We will give just a sketch of such a solution here. (Everything becomes evident after thorough consideration of the graphical representation). The inequalities can be rewritten as $1/a < x < 2/a$ and $2/a < x < 3/a$. Consider the vertical axis on the coordinate plane as x, and the horizontal axis as $1/a$. Draw three new lines: $x = 1/a$, $x = 2/a$, x=3/a. We see that values of $1/a$ belonging to the intervals $(2.5, 3)$ and $(3, 3.5]$, and also $1/a = 4$ are right for us. Thus it is possible to find 2,3 or 4 integer solutions.

4. Three children Ann, Borya and Vitya sit at the round table and eat nuts. Children have more than 3 nuts. At the beginning Ann owns all nuts. If Ann has even number of nuts, she divides them into two equal parts and gives to Borya and Vitya and if the number of her nuts is odd, then she eats 1 nut and then does the same. Then the next child (one by one, around the table) does the same: divides all his (or her) nuts between two others eating one nut in the process, if it is necessary. And so on. Prove that:

(a) at least 1 nut will be eaten,

(b) the children won't eat all nuts.

(M.N. Vyaliy)

(a) SOLUTION 1. (a) Assume that Ann has a nuts at the very beginning. Suppose that no nuts are eaten. Write down couple of first steps:

Observe that after n-th step one of the children has 0 nuts, while two other ones have $xa/2n$ and $ya/2n$, where x and y are odd numbers. This proposition can be easily proved. If it is true for the n-th step, then after the next one the amounts of nuts will be 0, $xa/2^{n+1}$ and $(2y+x)a/2^{n+1}$, where x and $2y+x$ are also odd, i.e. the statement is true again. As the statement holds after the first step, it also holds after the second one, thus after the third one also, and so on. But since after each

	Ann	Borya	Vitya
Beginning	a	0	0
1 step	0	$a/2$	$a/2$
2 step	$a/4$	0	$3a/4$
3 step	$5a/8$	$3a/8$	0

step each child has an integer amount of nuts, the number a should be divisible by 2^n for every integer n, which is impossible. Therefore, at least one nut will be eaten.

SOLUTION 2. Denote by a the number of nuts owned by the child who will divide next, and the number for the next person by b. If no nuts are eaten, after the next step $a' = b+a/2$ and $b' = a/2$. Observe that $|a' - 2b'| = \frac{1}{2}|a - 2b|$. This means that such difference after each step is divided by two but remains integer. This is impossible, so at least one nut will be eaten .

(b) If there are more than 3 nuts at any moment, the proposition is proved. Otherwise, consider the moment of time when the total amount of nuts is three for the first time. After any step exactly two persons own nuts and the one with the greater amount divides next. Consequently, when the total amount is three, the dividing person owns 2 nuts. So after next step the situation will be exactly the same, again with 3 nuts.

5. Peter has n^3 white $1 \times 1 \times 1$-cubes. He wants to make a $n \times n \times n$-cube using them, and he wants to make this cube totally white from the outside. What is the minimum number of sides of the cubes Vasya has to paint in black to prevent Peter from doing this?

   (a) n=2,

   (b) n=3

(R.G. Zhenodarov) Answers (a) 2 sides (b) 12 sides

(a) SOLUTION. Evidently, one painted side is not enough. But if we paint two opposite sides on one cube, one of them always will be on the outside.

(b) SOLUTION. It is enough to paint all sides of two cubes, because when constructing the $3 \times 3 \times 3$ cube we can fully hide only 1 cube. Now we will show how Peter can accomplish his task if there are 11 or less painted sides. In this case not more than one cube can be fully painted, not more than 5 can have more than 1 painted side. At first, choose the cube with maximum number of painted sides and put it into the center. There are no fully painted cubes among the remaining ones, thus all of them can be used for the center cube of the side. Now choose 6 more cubes with the greatest number of painted sides. Now there are no cubes with two or more painted sides left. Therefore, all other painted sides can also be easily hidden: one painted side of the cube can always be hidden.