International Mathematics
TOURNAMENT OF THE TOWNS

Solutions[1] A-level, Seniors **Fall, 2006**

1. COUNTEREXAMPLE. Consider a company: a host, his three sons and three guests. The guests do not know each other, the host knows all the guests, while each son knows only two guests. No two sons know the same pair of the guests. It is clear, that guests chords intersect the host chord in three distinct points; one point is between the others two. Further, this two guest chords lie on the different sides of the guest chord in between. Then the chord of the son who knows only these two guests must intersect the middle chord. Contradiction.

2. Consider triangle $A_1B_1C_1$. Let A_2 be intersection point of bisectors of exterior angles B_1 and C_1, while B_2 and C_2 be intersections of bisectors of exterior angles A_1 and C_1, and A_1 and B_1 respectively. Notice, that A_2 is equidistant from side B_1C_1, extension of side A_1B_1 and extension of side A_1C_1. Therefore, A_2 belongs to bisector A_1A; moreover, A_1A_2, B_1B_2, C_1C_2 are altitudes of triangle $A_2B_2C_2$. Let us prove that triangle $A_2B_2C_2$ and triangleABC coincide. Assume that A_2 is outside of triangle ABC. Note, that ray A_2B_2 intersects side AB of triangle ABB_1 at C_1 and does not intersect side AB_1 since sides AB and AB_1 are separated by A_2A_1. Therefore, B_2 is inside of triangle ABC. In the same way C_2 is inside of triangle ABC. However, segment B_2C_2 must intersect side BC at point A_1. Contradiction.

3. Let us assume that a is rational. Then a is periodic decimal fraction with period k. Then starting from some place the digits occupying the positions $k,\ 10k, \dots, 10^m k, \dots$ coincide. On the other hand, these are consecutive digits of representation $\sqrt{k}$. However, an irrational number cannot be represented by periodical fraction. Therefore, a is irrational.

4. ANSWER: no.

   The total sum of volumes of the pyramids with bases on the bottom base of the prism does not exceed one third of the prism volume. The same is true for the pyramids with bases on the top base of the prism. Therefore, the total sum of volumes of all the pyramids is less than the volume of the prism. Contradiction.

5. Let us consider $n = p(p-1) - 1$, where p is an odd prime number. Notice, that b_{n+1} is not divisible by p. Really, in the corresponding sum only denominators of the fractions $\frac{1}{p}, \frac{1}{2p}, \dots, \frac{1}{(p-1)p}$ are divisible by p.

   However, by regrouping the fractions in the following way:

$$\frac{1}{p} + \frac{1}{(p-1)p} = \frac{1}{(p-1)}, \quad \frac{1}{p} + \frac{1}{(p-2)p} = \frac{1}{2(p-2)}$$

   etc., we see that no factor of b_{n+1} is divisible by p.

   We have

$$\frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}} - \frac{1}{(p-1)p} = \frac{(a_{n+1}(p-1)p - b_{n+1})}{b_{n+1}(p-1)p}.$$

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Assuming that this fraction is reducible by factor d we get: $a_{n+1}(p-1)p = b_{n+1}(\text{mod } d)$, and $b_{n+1}(p-1)p = 0(\text{mod } d)$.

Then, $a_{n+1}(p-1)^2p^2{=}b_{n+1}(p-1)p \pmod d$. Note, that $\gcd(d,p) = 1$ (otherwise, b_{n+1} is divisible by p) and $(d, a_{n+1}) = 1$ (otherwise, b_{n+1} is divisible by their common divisor which implies that a_{n+1} and b_{n+1} share a common factor). Thus, $(p-1)^2$ is divisible by d. Therefore, $d \le (p-1)^2$.

Then,

$$b_n \ge \frac{b_{n+1}(p-1)p}{(p-1)^2} = \frac{b_{n+1}p}{(p-1)} > b_{n+1}.$$

Statement of the problem follows from the latter estimate and the fact that number of primes is infinite.

6. Consider a 4×13 board, where the rows correspond to the suits while the columns correspond to the values. A rook starts from left-bottom corner corresponding to Ace of Spade, visits each square of the board ones, and returns to the original square. (A rook can move either horizontally or vertically; it can jump through squares). It is clear that there is one-to-one correspondence between the number of the arrangements of a deck in a regular way and the number of rook circuits on this board. Let us code the circuits by placing the numbers from 1 to 52 in squares that rook visits on its way. For any circuit, the path starts and ends in square number 1; moreover, any two consecutive numbers are placed either at the same row or at the same column.

   a) Consider a circuit. Assume, that the first column is fixed. Notice, that if any two other columns trade places then we get a new circuit (different numeration of the table). Then by permuting 12 columns we get 12! of circuits that belong to the same group. Therefore, the number of circuits is a factor of 12!

   b) Let us prove that the number of circuits is also a factor of 13. Let us fold the board into a cylinder by joining its vertical sides. Any of 12 possible rotations of the cylinder transforms a given circuit to a new one that starts from square with the number different of 1. However, since the path still passes through the square with number 1, we may consider it as a regular circuit. Really, the corresponding numeration can be obtained by shifting all the numbers by the same value (modulo 52), so we get 1 at the left-bottom corner. Let us prove that new circuit is different from the original one. Assume that under some rotation a circuit transforms into itself. Let us consider any horizontal move (there must be one). Note, that 13 is a prime number. Thus, if we repeat this rotation 13 times then we would come to original point and each square of the horizontal would be visited; moreover, the only possible exit from any square of this horizontal is a horizontal one. This implies, that it is not possible to change a suit. Contradiction.

7. a) An inequality

$$4(x_1^2 + \cdots + x_k^2) < 2(x_1 + \cdots + x_k) < (x_1^3 + \cdots + x_k^3)$$

   implies that for at least one value (let it be x_1) we have $4x_1^2 < x_1^3$. Therefore, $x_1 > 4$. Then $(2x_2^2 - x_2) + \cdots + (2x_k^2 - x_k) < 4 - 2 \cdot 4^2 = -28$. Since the minimum of $2x^2 - x$ is $-1/8$, then $k - 1 > 8 \cdot 28 > 50$.

b) Consider, for example, $k = 2501, x_1 = 10, x_2 = x_3 = \cdots = x_{2501} = 0.1$. Then

$$\begin{aligned}
&x_1^2 + \cdots + x_{2501}^2 = 100 + 25 = 125,\\
&x_1 + \cdots + x_{2501} = 10 + 250 = 260,\\
&x_1^3 + \cdots + x_{2501}^3 > 1000.
\end{aligned}$$