International Mathematics
TOURNAMENT OF THE TOWNS

Solutions[1] A-level, Juniors **Fall, 2006**

1. Let $2a$ be the length of a side of a regular polygon, while r and R be the radii of its inscribed and circumscribed circles. Since the radius of the inscribed circle is perpendicular to the side of the polygon and touches it at its midpoint, then $a^2 + r^2 = R^2$. Therefore, the area of the ring between the circles is equal to $\pi(R^2 - r^2) = \pi a^2$. This implies the statement of the problem.

2. Counterexample. Consider a company: a host with three sons and three guests. The guests do not know each other, the host knows all the guests, while each son knows only two guests. No two sons know the same pair of the guests. It is clear, that the guests chords intersect the host chord in three distinct points; one point is between two others. So, the guest chord through this point separates two other guest chords. Therefore, the chord of the son who knows only two latter guests must intersect the guest chord in between. Contradiction.

3. a) Let S be a magic sum. Then
$$(a+b+c)+(a+d+g)+(c+f+i)+(g+h+i) = 4S = 2(b+e+h)+2(d+e+f). \quad (1)$$
Subtracting $(b+d+f+h)$ from both sides, we get $2(a+c+g+i){=}b+d+f+h+4e$.

   b) Let us notice that $a+i = c+g = b+h = d+f = S-e$. Combining with (1) we get $4(S-e) = 2(S-e)+4e$; therefore, $S = 3e$. Next, let us prove
$$2(a^2+c^2+g^2+i^2) = b^2+d^2+f^2+h^2+4e^2. \quad (2)$$
We have $a+c = S-b = h+e$, $c+i = S-f = d+e$, $g+i = S-h = b+e$, $a+g = S-d = f+e$. In addition, we have
$$ac+ci+ag+gi = (a+i)(c+g) = (S-e)^2 = 2e(S-e) = e(b+d+f+h).$$
Therefore,
$$\begin{aligned}&2(a^2+c^2+g^2+i^2) = \\ &(a+c)^2+(c+i)^2+(a+g)^2+(g+i)^2-2(ac+ci+ag+gi) = \\ &(h+e)^2+(d+e)^2+(f+e)^2+(b+e)^2-2e(b+d+f+h) = \\ &b^2+d^2+f^2+h^2+4e^2.\end{aligned}$$
To finish the proof let us notice that the statement of b) holds if we increase each entry of the table by the same value. Really,
$$\begin{aligned}&2((a+t)^3+(c+t)^3+(g+t)^3+(i+t)^3) = \\ &\quad 2((a^3+c^3+g^3+i^3)+3t(a^2+c^2+g^2+i^2)+3t^2(a+c+g+i)+4t^3) = \\ &b^3+d^3+f^3+h^3+4e^3+3t(b^2+d^2+f^2+h^2+4e^2)+3t^2(b+d+f+h+4e)+8t^3 = \\ &\qquad (b+t)^3+(d+t)^3+(f+t)^3+(h+t)^3+4(e+t)^3.\end{aligned}$$
Therefore, it is enough to consider the case $e = 0$. However, in this case the statement is obvious, since $a+i = c+g = b+h = d+f = 2e = 0$.

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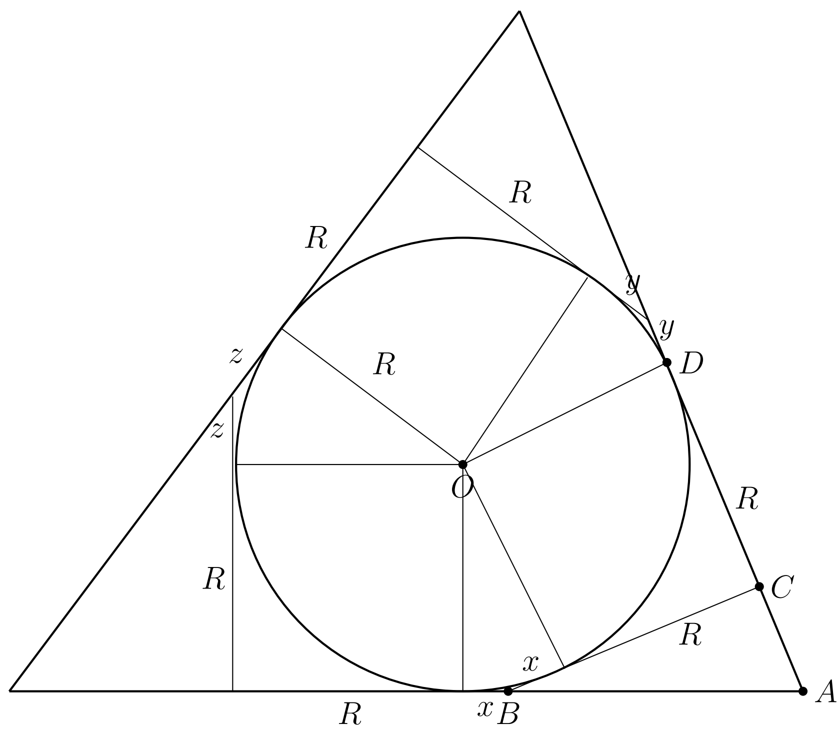
4. Let us notice that the hexagon constructed is split into six quadrilaterals by points of tangency of its inscribed circle. It is easy to see that three of them are squares with side R.

   Then, perimeter of the hexagon is $Q = 6R + 2x + 2y + 2z$ where x, y, and z are defined by a picture below. We use as well known fact that diameter of a circle inscribed into right angle triangle is equal to sum of of its legs minus hypothenuse. Let r_1 be radius of a circle inscribed into right angle triangle ABC.

   Then $2r_1 = AC + BC - AB = (AD - R) + (R + x) - (AF - x) = 2x + (AD - AF) = 2x$. In similar way, we find that $2y$ and $2z$ are the diameters of the two other circles. Then the sum in question equals $2x + 2y + 2z = Q - 6R$.



5. Look at next page

6 Let us consider $n = p(p-1) - 1$, where p is an odd prime number. Notice, that b_{n+1} is not divisible by p. Really, in the corresponding sum of fractions only denominators of the fractions $\frac{1}{p}, \frac{1}{2p}, \dots, \frac{1}{(p-1)p}$ are divisible by p.

   However, by regrouping the fractions

$$\frac{1}{p} + \frac{1}{(p-1)p} = \frac{1}{(p-1)}, \quad \frac{1}{p} + \frac{1}{(p-2)p} = \frac{1}{2(p-2)}$$

   etc., we see that no factor of b_{n+1} is divisible by p.

   We have

$$\frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}} - \frac{1}{(p-1)p} = \frac{(a_{n+1}(p-1)p - b_{n+1})}{b_{n+1}(p-1)p}.$$

   Assuming that this fraction is reducible by factor d we get: $a_{n+1}(p-1)p = b_{n+1} \mod d$, and $b_{n+1}(p-1)p = 0 \mod d$.

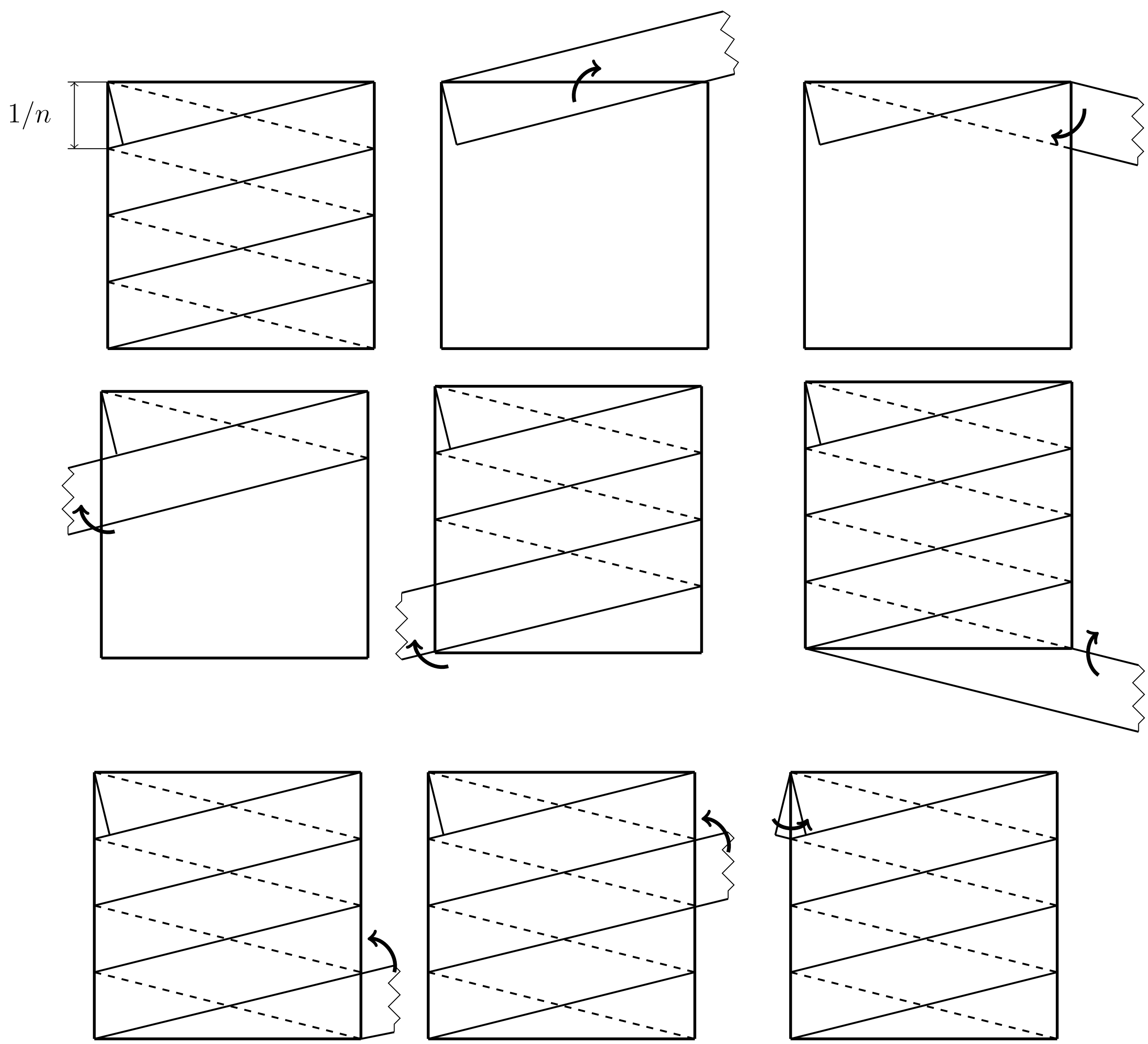
Then, $a_{n+1}(p-1)^2p^2 \equiv b_{n+1}(p-1)p \pmod{d}$. Note, that $\gcd(d,p)=1$ (otherwise, b_{n+1} is divisible by p) and $(d, a_{n+1})=1$ (otherwise, b_{n+1} is divisible by their common divisor which implies that a_{n+1} and b_{n+1} share a common factor). Thus, $(p-1)^2$ is divisible by d. Therefore, $d \le (p-1)^2$.

Then,

$$b_n \ge \frac{b_{n+1}(p-1)p}{(p-1)^2} = \frac{b_{n+1}p}{(p-1)} > b_{n+1}.$$

Statement of the problem follows from the latter estimate and the fact that number of primes is infinite.

5 b) Consider a set of rectangles with sides $1/\sqrt{n^2+1}$ and $2\sqrt{n^2+1}$. See the picture below to learn how to wrap a square with these rectangles.



7 Answer: 34.

Solution. In his first question, spectator (S) calls the top and bottom cards. The answer "50" reveals two outmost cards. Let us number either of them by 1 and the other by 52, defining the order of the deck. Then, S calls the pair $(1, 3)$. The answer "1" reveals the card numbered 3 (S intentionally skips card 2, which we refer to as the "space"). He continues to ask questions in pairs. In odd questions he calls two farthest unmentioned cards (one of them is "space", we refer to the other as the "antispace"), reassigns unmentioned card adjacent to "antispace" as new "space" and in his next (even) question he calls two cards adjacent to "space".

Thus, in the second pair of the questions, S calls the pairs $(2, 51)$ (two farthest unmentioned cards) and $(51, 49)$ (two cards adjacent to "space"). Next pair of questions is $(50, 4), (4, 6)$, then $(5, 48), (48, 46)$ and so on. After the first pair of questions, the audience is aware that the situation is the following (revealed cards are boxed):

$$\boxed{1}\quad 2\quad \boxed{3}\quad 4\quad 5\quad 6\quad \dots\quad 48\quad 49\quad 50\quad 51\quad \boxed{52}$$

Notice that while the first pair of questions reveals all three of mentioned cards, the second pair of questions would leave two possible arrangements. We refer to the actual arrangement of cards as the "main case" and to the other possible arrangement of cards as the "auxiliary case".

So, after the second pair of questions, the situation is one of the following (cards that have been mentioned but not yet revealed are underlined):

$$\boxed{1}\quad \underline{2}\quad \boxed{3}\quad 4\quad 5\quad 6\quad \dots\quad 48\quad \underline{49}\quad 50\quad \underline{51}\quad \boxed{52}\ \text{(main case)}$$

$$\boxed{1}\quad \underline{2}\quad \boxed{3}\quad \underline{4}\quad 5\quad 6\quad \dots\quad 48\quad 49\quad 50\quad \underline{51}\quad \boxed{52}\ \text{(auxiliary case)}$$

To distinguish the main and auxiliary cases, we observe that the distance between the two farthest unmentioned cards is different (it is lower in the auxiliary case). The answer on next question would eliminate the auxiliary case.

Really, the fifth question names cards $(50, 4)$. The answer "45" leaves the main case; otherwise, the answer would be "44".

Thus, after 5 questions (in total) we come to the following situation:

$$\boxed{1}\quad \boxed{2}\quad \boxed{3}\quad \underline{4}\quad 5\quad 6\quad \dots\quad 48\quad \boxed{49}\quad \underline{50}\quad \boxed{51}\quad \boxed{52}$$

One may check that after 33 questions we get:

$$\boxed{1}\quad \dots\quad \boxed{24}\quad \underline{25}\quad 26\quad 27\quad \boxed{28}\quad \underline{29}\quad \boxed{30}\quad \dots\quad \boxed{52}$$

In his last question S calls $(25, 26)$. The answer 0 eliminates an auxiliary case, and (27) card is revealed as the last card left.

Now let us show that 33 (or less) questions are not enough. Let us assume that originally all the cards are split into 52 groups; one card in each group. In case of a question when named cards belong to different groups, we combine these groups into one. So, each question decreases the number of groups maximum by one. Therefore, after 33 questions the number of groups left is no less than $52 - 33 = 19$. Among them the number of groups consisting of at least 3 cards is no more than 17. Thus, there are either two groups consisting of one card or there is a group consisting of exactly 2 cards. In either case if these two cards trade places, while the rest of the cards remain untouched, then the answers will be the same. This means that the order of cards can not be restored uniquely.