Junior A-Level Paper[1]   **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. The altitudes AD and BE of triangle ABC meet at its orthocentre H. The midpoints of AB and CH are X and Y, respectively. Prove that XY is perpendicular to DE.

3. Baron Münchhausen's watch works properly, but has no markings on its face. The hour, minute and second hands have distinct lengths, and they move uniformly. The Baron claims that since none of the mutual positions of the hands is repeats twice in the period between 8:00 and 19:59, he can use his watch to tell the time during the day. Is his assertion true?

4. A 10×12 paper rectangle is folded along the grid lines several times, forming a thick 1×1 square. How many pieces of paper can one possibly get by cutting this square along the segment connecting

   (a) the midpoints of a pair of opposite sides;
   (b) the midpoints of a pair of adjacent sides?

5. In a rectangular box are a number of rectangular blocks, not necessarily identical to one another. Each block has one of its dimensions reduced. Is it always possible to pack these blocks in a smaller rectangular box, with the sides of the blocks parallel to the sides of the box?

6. John and James wish to divide 25 coins, of denominations 1, 2, 3, ..., 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

7. The squares of a chessboard are numbered in the following way. The upper left corner is numbered 1. The two squares on the next diagonal from top-right to bottom-left are numbered 2 and 3. The three squares on the next diagonal are numbered 4, 5 and 6, and so on. The two squares on the second-to-last diagonal are numbered 62 and 63, and the lower right corner is numbered 64. Peter puts eight pebbles on the squares of the chessboard in such a way that there is exactly one pebble in each column and each row. Then he moves each pebble to a square with a number greater than that of the original square. Can it happen that there is still exactly one pebble in each column and each row?

Note: The problems are worth 4, 5, 5, 2+4, 6, 6 and 8 points respectively.

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[1]Courtesy of Andy Liu.

Solution to Junior A-Level Spring 2005

1. Let $f(x)$ be a polynomial with integral coefficients such that $f(x_1)$ and $f(x_2)$ are integers for some integers x_1 and x_2. Since $x_1^k - x_2^k$ is divisible by $x_1 - x_2$ for all k, $f(x_1) - f(x_2) = n(x_1 - x_2)$ for some integer n. If in addition the distance between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is also an integer m, then $(x_1 - x_2)^2 + (f(x_1) - f(x_2))^2 = m^2$. Then $(x_1 - x_2)^2(1 + n^2) = m^2$, so that $1 + n^2$ is also the square of an integer. This is only possible for $n = 0$. Hence $f(x_1) - f(x_2) = 0$, so that $f(x_1) = f(x_2)$, and the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is indeed parallel to the x-axis.

2. Since $\angle ADB = 90° = \angle AEB$, D and E lie on a circle with diameter AB, and hence with centre X. Since $\angle CDH = 90° = \angle CEH$, D and E lie on a circle with diameter CH, and hence with centre Y. The common chord DE of the two circles is therefore perpendicular to the line of centres XY.
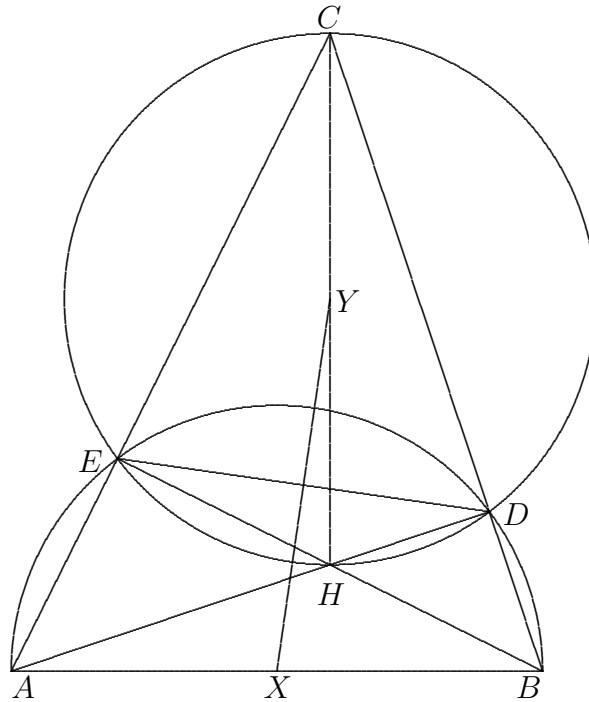
3. We first show that the three hands coincide only at 12:00 or 24:00. Suppose this occurs again. Consider the angular distance θ covered by the hour hand where $0° < \theta < 360°$. The angular distance covered by the minute hand is $360°n + \theta$, where n is the number of revolutions it has made. Since the minute hand moves at 12 times the speed of the hour hand, $360°n + \theta = 12\theta$, so that $\theta = 360° \frac{n}{11}$. The angular distance covered by the second hand is $360°m + \theta$, where m is the number of revolutions it has made. Since the second hand moves at 720 times the speed of the hour hand, $360°m + \theta = 720\theta$, so that $\theta = 360° \frac{m}{719}$. From $\frac{n}{11} = \frac{m}{719}$, n must be a multiple of 11 and m a mutliple of 719 as 11 and 719 are relatively prime. However, this contradicts $0° < \theta < 360°$. This justifies the Baron's claim. If there are two indistinguishable times within a twelve-hour period, shift the times so that one of them is at 12:00 or 24:00 and the other not. However, since one set of hands coincide, so must the other, and we have already proved that this is not possible.

4. (a) Let the edge of length 12 be horizontal. No matter how the piece of paper is folded into a 1×1 stack, the horizontal edges of each square remains horizontal. Thus if the cut is horizontal, we obtain 10+1=11 strips of paper. If the cut is vertical, we obtain 12+1=13 strips of paper.

   (b) Label the vertices of the 1×1 squares as follows. Along the top row, they are labelled alternately A and B. Along the second row, they are labelled alternately C and D. Thereafter, the rows are labelled alternately as above, so that along the bottom row, the vertices are labelled alternately A and B. There are $6 \times 7 = 42$ A vertices, $6 \times 6 = 36$ B vertices, $5 \times 7 = 35$ C vertices and $5 \times 6 = 30$ D vertices. No matter how the piece of paper is folded into a 1×1 stack, all A vertices will be on top of one another, as will all the B vertices, all the C vertices and all the D vertices. If the cut isolates the A vertices, we have 42+1=43 pieces of paper. If the cut isolates the B vertices, we have 36+1=37 pieces of paper. If the cut isolates the C vertices, we have 35+1=36 pieces of paper. If the cut isolates the D vertices, we have 30+1=31 pieces of paper.

5. Let the box be $10000 \times 1000 \times 100$, the first block 10000×1000, the second block $10000 \times 800 \times 80$ and the third block $10000 \times 200 \times 80$. Let the reduced first block be $10000 \times 999 \times 20$, the reduced second block be $10000 \times 800 \times 79$ and the reduced third block be $9999 \times 200 \times 80$. The dimension of the box which exceeds 1000 must remain at 10000. The dimension of the box which exceeds 100 must remain 1000=800+200, and the third dimension of the box must remain 100=80+20. Hence no smaller box can hold the three reduced blocks.

6. James can always get more kopeks than John. Upon John's initial offer, James can either take it or leave it. If there is a way for him to get more kopeks than John by taking it, there is nothing further to prove. If there are no ways, then he makes John take it, and there are no ways for John to get more kopeks than he.

7. Label the rows from 1 to 8 from top to bottom, and the columns from 1 to 8 from left to right. Note that the sum of the row number and column number of a square is constant along any diagonal from top-right to bottom-left, and this sum increases as the diagonals shift from top-left to bottom-right. For eight pebbles each in a different row and a different column, the sum of their row and column numbers must be $2(1+2+3+4+5+6+7+8)$. In moving a pebble from a square to another so that the number on the square increases, it must either slide downwards along a diagonal from top-right to bottom-left, or move to a diagonal closer to the bottom-right. Since the sum of all the row and column numbers cannot decrease, every pebble must stay on its original diagonal from top-right to bottom-left. However, this means that every pebble slides downwards, so that there will not be any left in the first row.