International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2005.**[1]

1. Do there exist positive integers $a,\ b,\ n$ such that $n^2 < a^3 < b^3 < (n+1)^2$?
2. A segment of length $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is given. Can a segment of length 1 be constructed using a straight-edge and a compass?
3. One of 6 coins is a fake. We do not know the weight of either a real coin or the fake coin, except that the eal coins all weigh the same, but different from the fake coin. Using a scale which shows the total weight of the coins being weighed, how can the fake coin be found in 3 weighings?
4. On all three sides of a right triangle ABC, external squares are constructed, their centres being $D,\ E$ and F. Prove that the ratio of the area of triangle DEF to the area of triangle ABC is
   (a) greater than 1;
   (b) at least 2.
5. A cube lies on the plane, with a letter A on its top face. In each move, it is rolled over one of its bottom edges onto the adjacent face. After a few moves, the cube returns to its initial position, again with the letter A on its top face. Is it possible for the letter A to have made a 90° turn?

Note: The problems are worth 3, 3, 4, 2+2 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

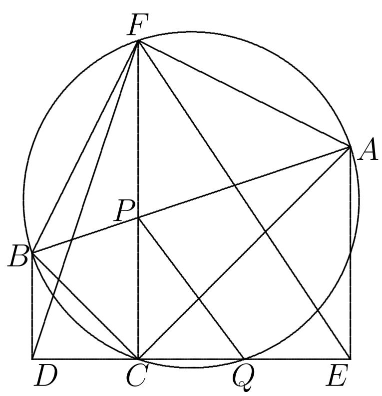
Solution to Senior O-Level Fall 2005

1. Suppose such integers a, b and n exist. Since $a+1 \le b$, $n^2 < a^3 < (a+1)^3 < (n+1)^2$. Note that $n^2 < a^3 < a^4$, so that $n < a^2$. Hence $(a+1)^3 > a^3 + 3a^2 + 1 > n^2 + 2n + 1 = (n+1)^2$, which is a contradiction.

2. Construct any segement of length x. Construct a right triangle with legs x and x. Then the hypotenuse will have length $\sqrt{2}x$. Construct a right triangle with legs x and $\sqrt{2}x$. Then the hypotenuse will have length $\sqrt{3}x$. Construct a right triangle with legs $\sqrt{2}x$ and $\sqrt{3}x$. Then the hypotenuse will have length $\sqrt{5}x$. Construct a segment PQ of length $(\sqrt{2}+\sqrt{3}+\sqrt{5})x$ and extend it to R such that $QR = x$. On another ray from P, mark off the point S where PS has the given length $\sqrt{2}+\sqrt{3}+\sqrt{5}$. Through R, draw the line parallel to QS, cutting the ray at T. Then triangles PQS and PRT are similar, so that $\frac{ST}{PS} = \frac{QR}{PQ}$. It follows that $ST = 1$.

3. Let the coins be A, B, C, D, E and F. In three weighings, we determine the average weight m of C and E, the average weight n of D and F, and the average weight k of B, E and F. If $m = n = k$, the fake coin is A. If $m = n \ne k$, the fake coin is B. If $m \ne n = k$, the fake coin is C. If $k = m \ne n$, the fake coin is D. If $k \ne m \ne n \ne k$, then the fake coin is E or F. This can be distinguished since $2m + n = 3k$ if it is E, and $m + 2n = 3k$ if it is F.

4. Denote the area of triangle T by $[T]$. Since $\angle BCD + \angle BCA + \angle ACE = 45° + 90° + 45° = 180°$, C lies on DE. Since $\angle BCA + \angle BFA = 90° + 90° = 180°$, $BCAF$ is a cyclic quadrilateral. Hence $\angle FCD = \angle FCB + \angle BCD = \angle FAB + \angle BCD = 45° + 45° = 90°$. Let P be the point of intersection of FC and AB.



   (a) Since BD and AE are perpendicular to DE, they are parallel to CF. Since $CF > CP$, $[DCF] > [BCP]$ and $[ECF] > [ACP]$. Hence $[DEF] > [ABC]$.

   (b) Let K be the other point of intersection of DE with the circumcircle of triangle ABC. Then FQ is a diameter of the circle. Hence $PF = PQ \ge PC$ so that $CF \ge 2CP$. It follows that $[DCF] \ge 2[BCP]$ and $[ECF] \ge 2[ACP]$, so that $[DEF] \ge 2[ABC]$.

5. The vertices of a cubic lattice may be painted black and white such that no two vertices of the same colour are adjacent. The vertices of the cube are painted in the same colours as the vertices of its initial position in the cubic lattice. When the cube is rolled over, its white vertices always go to white vertices of the cubic lattice, and its black vertices always go to black vertices of the cubic lattice. When it returns to its initial position, again with the letter A on its top face, the letter A cannot have made a 90° turn as this requires the vertices of the cube to have different colours from the corresponding vertices of the cubic lattice.