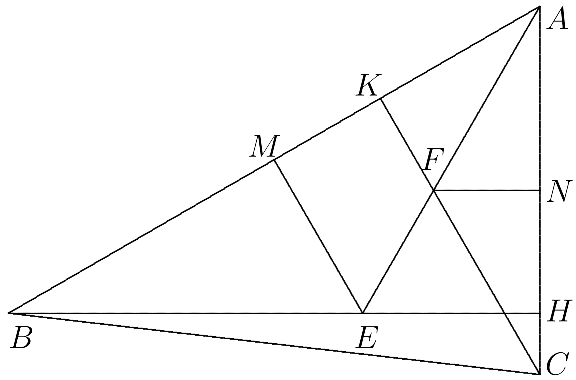
International Mathematics
TOURNAMENT OF THE TOWNS

Solution to Junior O-Level Spring 2004[1]

1. Let M and N be the respective midpoints of AB and AC. Let the extension of BE cut AC at H, and the extension of CF cut AB at K. Note that triangles AEH, AEM and BEM are congruent to one another. Hence $\angle BEM = \angle MEA = \angle AEH = 60°$. It follows that $\angle MAE = \angle EAH = 30°$. Since triangles AFN and CFN are congruent to each other, $\angle FCN = 30°$, so that $\angle CKA = 90°$. Thus CF is indeed perpendicular to AB.



2. Clearly, we can have $n = 1$ by taking any prime number. We can also have $n = 2$ since each odd prime is the sum of two consecutive numbers. Suppose $p = a + (a + 1) + \cdots + (a + k)$ for some prime number p and positive integers a and $k \geq 2$. Then $2p = (k + 1)(2a + k)$. Each of $k + 1$ and $2a + k$ is greater than 2. This is a contradiciton since p is a prime number. Hence $n = 1$ or 2.

3. (a) We describe the process in the following chart.

Action Taken	Amount in Bucket A	Bucket B	Bucket C
Initial State	3	20	0
Pour from B into C until C=A	3	17	3
Pour away C	3	17	0
Pour from B into C until C=A	3	14	3
Pour away C	3	14	0
Pour from B into C until C=A	3	11	3
Pour away C	3	11	0
Pour from B into C until C=A	3	8	3
Pour away C	3	8	0
Pour from B into C until C=A	3	5	3
Pour from A into C until C=B	1	5	5
Pour from B into A until A=C	5	1	5
Pour from C into A	10	1	0

[1]Courtesy of Andy Liu.

(b) If $n \equiv 0 \pmod 3$, the task is impossible, because the amount of liquid in any bucket at any time will be a multiple of 3, but our target 10 is not. Suppose $n \equiv 2 \pmod 3$. If $n = 2$ or 5, we do not have enough water. If $n = 8$, we can proceed as in (a) from the partition line in the chart. If $n \geq 11$, we can reduce the amount 3 litres at a time. Finally, suppose $n \equiv 1 \pmod 3$. If $n = 1$ or 4, we do not have enough water. If $n = 7$, we can simply pour everything from bucket B into bucket A. If $n \geq 10$, we can reduce the amount 3 litres at a time. In summary, the task is possible except for $n = 1, 2, 4, 5$ and $n \equiv 0 \pmod 3$.

4. Note that $b = a(10^n + 1)$ so that $\frac{b}{a^2} = \frac{10^n+1}{a}$. Suppose it is an integer d. Since a is an n-digit number, $1 < d < 11$. Since $10^n + 1$ is not divisible by 2, 3 or 5, the only possible value for d is 7. The example $a = 143$ and $b = 143143$ shows that we can indeed have $d = 7$.

5. There are 9×10^9 10-digit numbers. If two of them are non-neighbours, they cannot have the same digits in each of the first nine places. Thus the number of 10-digit numbers we can choose is no more than the number of 9-digit numbers, which is 9×10^8. On the other hand, for each 9-digit number, we can add a unique tenth digit so that the sum of all 10 digits is a multiple of 10. If two of the 10-digit numbers obtained this way differ in only one digit, not both digit sums can be multiples of 10. Hence no two are neighbours among these 9×10^8 10-digit numbers.