Seniors
(Grades 11 & 12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2004.**[1]

1 [5] Functions $f(x)$ and $g(y)$, defined for all real x and y satisfy conditions

$$f\big(g(y)\big) = y, \qquad g\big(f(x)\big) = x \qquad \text{for all } x \text{ and } y.$$

It is known that $f(x) = kx + h(x)$, where k is a coefficient and $h(x)$ is a periodic function. Prove that $g(y)$ is also a sum of a linear and a periodic function.

Function h is called *periodic* if there exists $d \neq 0$ such that $h(x + d) = h(x)$ for all x.

2 [5] In turns Joe and Pete pick up pebbles from the pile. Joe starts. On his turn he takes either 1 or 10 pebbles. On his turn Pete takes either m or n pebbles.

The player who cannot move, loses. It is known that Joe has a winning strategy for any initial number of pebbles in the pile (he can win no matter how Pete plays). Find possible values of m and n.

3 [5] The results of operations

$$x + y, \qquad x - y, \qquad xy, \qquad x/y$$

are written on four cards that are placed on a table in random order. Prove that one can restore both x and y given that x and y are positive numbers.

4 [6] A circle with the center I is entirely inside of a circle with center O. Consider all possible chords AB of the larger circle which are tangent to the smaller one. Find the locus of the centers of the circles circumscribed about the triangle AIB.

5 [7] here are two rectangles A and B. It is known that one can tile a rectangle similar to B using copies of A. Prove that one can tile a rectangle similar to A using copies of B.

6 [8] Let n be an integer divisible by neither 2 nor 3. Let us call a triangle *admissible* if all its angles are in the form $\frac{m}{n} \cdot 180^\circ$ where m is an integer. Triangles which are not similar we call *essentially different*.

In the beginning there is one admissible triangle. The following procedure is applied: we chose a triangle from the the set obtained on the previous stage and cut it into two admissible triangles so that all the triangles in the new set are essentially different.

This procedure repeats itself until it is possible. Prove that in the end we get all possible admissible triangles.

7 [8] Let $\angle AOB$ be obtained from $\angle COD$ by rotation (ray AO transforms into ray CO). Let E and F be the points of intersection of the circles inscribed into these angles.

Prove that $\angle AOE = \angle DOF$.

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[1]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].