International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Fall 2004.**

1. An angle is said to be rational if its measure in degrees is a rational number. A triangle is said to be rational if all its angles are rational. Prove that there exist at least three different points inside any acute rational triangle such that when each is connected to the three vertices of the original triangle, we obtain three rational triangles.

2. The incircle of triangle ABC touches the sides BC, CA and AB at D, E and F respectively. If $AD = BE = CF$, does it follow that ABC is equilateral?

3. What is the maximum number of knights that can be place on an 8×8 chessboard such that each attacks at most seven other knights?

4. On a blackboard are written four numbers. They are the values, in some order, of $x + y$, $x - y$, xy and $\frac{x}{y}$ where x and y are positive numbers. Prove that x and y are uniquely determined.

5. K is a point on the side BC of triangle ABC. The incircle of triangle BAK touches BC at M. The incircle of triangle CAK touches BC at N. Prove that $BM \cdot CN > KM \cdot KN$.

6. Two persons share a block of cheese as follows. They take turns cutting an existing block of cheese into two, until there are five blocks. Then they take turns choosing one block at a time. The person who makes the first cut also makes the first choice, and gets an extra block. Each wants to get as much cheese as possible. What is the optimal strategy for each, and how much is each guaranteed to get, regardless of the counter measures of the other?

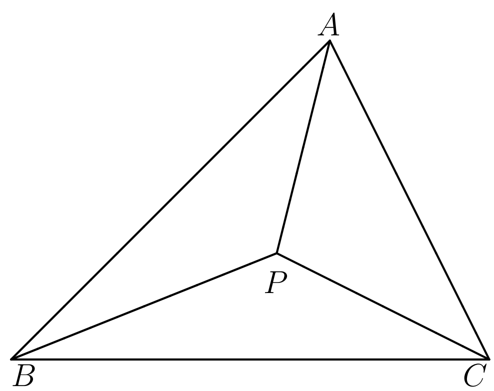
7. We have many copies of each of two rectangles. If a rectangle similar to the first can be made by putting together copies of the second, prove that a rectangle similar to the second can be made by putting together copies of the first, with no overlapping in both instances.

Note: The problems are worth 4, 5, 6, 6, 7, 8 and 8 points respectively.

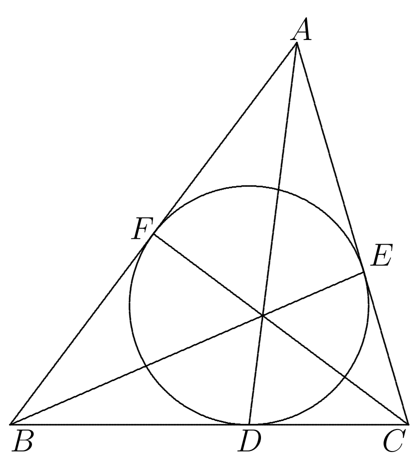
[1] Courtesy of Andy Liu.

Solution to Junior A-Level Fall 2004

1. We remark that if a triangle has two rational angles, then the third angle must also be rational, so that the triangle is rational. Let P be a point inside an acute rational triangle ABC, which is then divided into triangles PBC, PCA and PAB. In each of the following cases, it is sufficient to prove that PBC is a rational triangle, since we can prove that PCA and PAB are also rational in an analogous manner. Suppose P is the incentre of triangle ABC. Since PB bisects the $\angle ABC$ and CP bisects $\angle BCA$, both of which are rational, both $\angle PBC$ and $\angle PCB$ are rational. Hence triangle PBC is rational. Suppose P is the circumcentre of triangle ABC. Since ABC is acute, P is indeed inside it. Now $\angle CPB = 2\angle CAB$ is rational, so that $\angle PCB = \frac{1}{2}(180° - \angle CPB)$ is also rational. Hence triangle PBC is rational. Suppose P is the orthocentre of triangle ABC. Since ABC is acute, P is indeed inside it. Now $\angle PBC = 90° - \angle BCA$ and $\angle PCB = 90° - \angle ABC$ are both rational. Hence triangle PBC is rational. If ABC is not equilateral, then its incentre, circumcentre and orthocentre are disinct points. Thus we have the required three points. If ABC is equilateral, there exist infinitely many points P on the perpendicular bisector of BC such that $\angle PCB$ is rational. Any three such points will meet the requirement of the problem.



2. Assume that $BE = CF$ but $AB \neq AC$. In triangles ABE and ACF, $\angle BAE = \angle CAF$, $AE = AF$ and $BE = CF$. Since $AB \neq AC$, ABE and ACF are not congruent triangles. Hence $\angle ABE \neq \angle ACF$ but we do have $\angle ABE + \angle ACF = 180°$. Hence either $\angle ABE$ or $\angle ACF$ is obtuse, which means that either $AE > AB$ or $AF > AC$. Since $AE = AF$, either $AE > AC$ or $AF > AB$. This is a contradiction. It follows that $AB = AC$, and we can prove in a similar way that $AD = CF$ implies $BC = BA$, so that ABC is indeed equilateral if $AD = BE = CF$.



3. Let us start with a knight on each square of the 8×8 chessboard. If we remove the 4 knights in the central 2×2 subboard, we are left with 60 knights each of which attacks at most 7 others. We now show that 60 is indeed the maximum. Again, we start with a knight on each of the 64 squares. Note that a knight can attack 8 other knights only if it occupies one of the squares in the central 4×4 subboard. We put these 16 knights on a black list. In the following diagram, the number on each square shows the maximum number of knights on the black list that can attack that square. Note that all the numbers are 4 or less. Thus the removal of a knight can take at most 4 other knights off the black list. Even if the removed knight itself is on the black list, we can take at most 5 knights off. Hence removing at most 3 knights will not clear the black list.

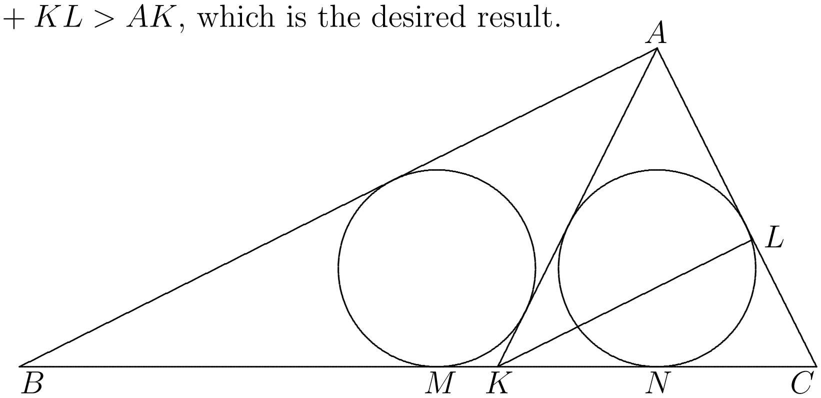
0	1	1	2	2	1	1	0
1	2	2	3	3	2	2	1
1	2	2	3	3	2	2	1
2	3	3	4	4	3	3	2
2	3	3	4	4	3	3	2
1	2	2	3	3	2	2	1
1	2	2	3	3	2	2	1
0	1	1	2	2	1	1	0

4. Note that $(x+y)+(x-y)=2x$ while $(xy)(\frac{x}{y})=x^2$, and that only $x-y$ can be non-positive. We consider three cases.
Case 1. All four numbers are positive.
Let a, b, c and d denote $x+y$, $x-y$, xy and $\frac{x}{y}$ in some order. Choose a pair of them and check if the square of their sum is four times the product of the other two numbers. The pair can be chosen in six ways. There are three subcases.
Subcase 1a. This is satisfied by two disjoint pairs.
We may assume that we have $(a+b)^2 = 4cd$ and $(c+d)^2 = 4ab$. Adding these two equations yields $(a-b)^2 + (c-d)^2 = 0$ so that $a = b$ and $c = d$. Substituting back into $(a+b)^2 = 4cd$, we have $a = \pm c$. Since all four numbers are positive, we must have $a = b = c = d$. This is a contradiction since $x + y \neq x - y$.
Subcase 1b. This is satisfied by two intersecting pairs.
We may assume that we have $(a+b)^2 = 4cd$ and $(a+c)^2 = 4bd$ with $b \neq c$. Then we have $b(a+b)^2 = 4bcd = c(a+c)^2$, or equivalently $(b-c)(a^2+2a(b+c)+(b^2+bc+c^2)) = 0$. This is a contradiction since $b - c \neq 0$ while $a^2 + 2a(b+c)(b^2+bc+c^2) > 0$.
Subcase 1c. This is satisfied by only one pair.
We may assume that $(a+b)^2 = 4cd$. Then we know that the larger one of a and b is $x+y$ and the smaller one $x-y$. We can determine x and y uniquely.
Case 2. One of the numbers is 0. We know that $x = y$ so that $\frac{x}{y} = 1$ must also be among the four numbers. The other two are $x + y = 2x$ and $xy = x^2$. Since their product is $2x^3$, we can determine $x = y$ uniquely.

Case 3. One of the numbers is negative.
We know that $x < y$ and $\frac{x}{y} < 1$. Check how many numbers in $S = \{x + y, xy, \frac{x}{y}\}$ lie strictly between 0 and 1. There are three subcases.
Subcase 3a. There is exactly one such number.
We know that this number is $\frac{x}{y}$, and we can determine x and y uniquely from $x - y$ and $\frac{x}{y}$.
Subcase 3b. There are exactly two such numbers.
We cannot have $x+y < 1$. Otherwise, we must have $x < 1$ and $y < 1$ so that $xy < 1$, but then all three numbers in S lie strictly between 0 and 1. Hence $x + y > 1$ is the largest number in S, and we can determine x and y uniquely from $x - y$ and $x + y$.
Subcase 3c. There are exactly three such numbers.
From $x + y < 1$, we have $x < 1$ and $y < 1$ so that $xy < x + y$ and $xy < \frac{x}{y}$. Hence the smallest number in S is xy, and we can determine x and y uniquely from $x - y$ and xy.

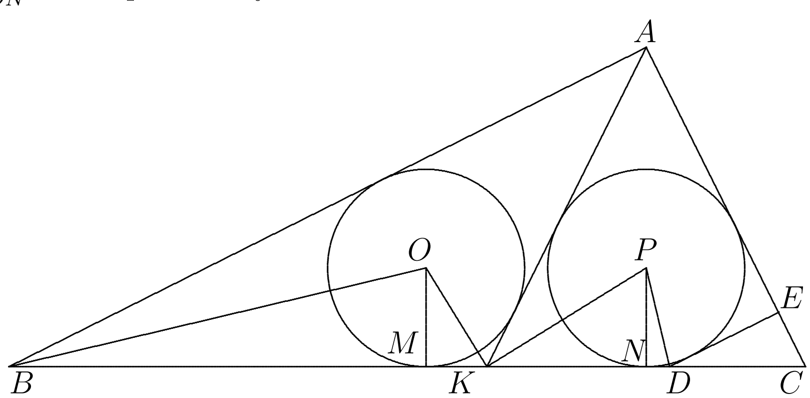
5. **First Solution:**

Note that $BM = \frac{AB+BK-AK}{2}$, $CN = \frac{AC+CK-AK}{2}$, $KM = \frac{AK+BK-AB}{2}$ and $KN = \frac{AK+CK-AC}{2}$. Hence $BM{\cdot}CN > KM{\cdot}KN$ is equivalent to $AC{\cdot}KB+AB{\cdot}KC > AK(KB+KC) = AK{\cdot}BC$, or $\frac{AC \cdot KB}{BC} + \frac{AB \cdot KC}{BC} > AK$. Let L be the point on AC such that KL is parallel to AB. Then triangles ABC and LKC are similar. Hence $AL = \frac{AC \cdot KB}{BC}$ and $KL = \frac{AB \cdot KC}{BC}$. By the Triangle Inequality, $AL + KL > AK$, which is the desired result.



Second Solution:

Construct the tangent to the incircle of triangle CAK parallel to AB and closer to C than to A, cutting BC at D and CA at E. Let O and P be the respective incentres of triangles BAK and CAK. Note that OK is perpendicular to PK since they bisect $\angle BKA$ and $\angle AKC$ respectively. Hence triangles MKO and NPK are similar, so that $\frac{OM}{KM} = \frac{KN}{PN}$. Since AB is parallel to DE, $\angle ABD + \angle BDE = 180°$. They are bisected respectively by OB and PD, which are thus perpendicular to each other. Hence triangle MOB is similar to triangle NDP, so that $\frac{BM}{OM} = \frac{PN}{DN}$. Multiplication yields $KM \cdot KN = BM \cdot DN < BM \cdot CN$.



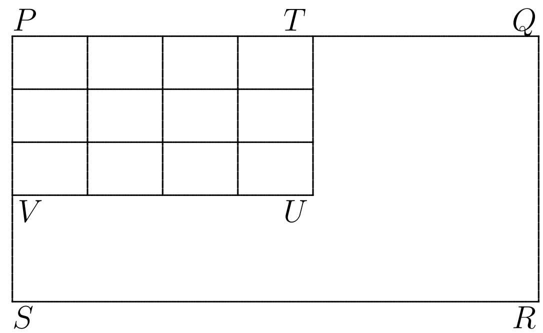
6. Let the total amount of cheese be 1, the first player be Alexei and the second player be Boris. Alexei can be assured of getting at least $\frac{3}{5}$ if he cuts 1 into $\frac{3}{5}$ and $\frac{2}{5}$. We consider two cases.
Case 1. Boris cuts $\frac{2}{5}$ into x and $\frac{2}{5}-x$, where $0 \le x \le \frac{1}{5}$.
Alexei cuts $\frac{3}{5}$ into x and $\frac{3}{5}-x$. Now the four pieces are of sizes $x = x \le \frac{2}{5}-x < \frac{3}{5}-x$. No matter how Boris makes his second cut, the second smallest piece is at most x, and the second largest piece is at most $\frac{2}{5}-x$ since $2(\frac{2}{5}-x) \ge \frac{3}{5}-x$. Hence Boris can get at most $x+(\frac{2}{5}-x)=\frac{2}{5}$.
Case 2. Boris cuts $\frac{3}{5}$ into x and $\frac{3}{5}-x$, where $0 \le x \le \frac{3}{10}$.
If $0 \le x \le \frac{1}{5}$, Alexei cuts $\frac{2}{5}$ into x and $\frac{2}{5}-x$, and this is the same as in Case 1. Hence we may assume that $\frac{1}{5} < x \le \frac{3}{10}$. Alexei cuts $\frac{3}{5}-x$ into $\frac{2}{5}-x$ and $\frac{1}{5}$. Now the four pieces are of sizes $\frac{2}{5}-x < \frac{1}{5} < x < \frac{2}{5}$. There are four subcases.
Subcase 2a. Boris cuts $\frac{2}{5}$ into y and $\frac{2}{5}-y$, where $0 \le y \le \frac{1}{5}$.
We have either $y \le \frac{2}{5}-x < \frac{1}{5} < x \le \frac{2}{5}-y$, in which case Boris gets $(\frac{2}{5}-x)+x=\frac{2}{5}$, or $\frac{2}{5}-x \le y \le \frac{1}{5} \le \frac{2}{5}-y \le x$, in which case Boris still gets $y+(\frac{2}{5}-y)=\frac{2}{5}$.
Subcase 2b. Boris cuts x.
If $\frac{1}{5}$ remains the third largest piece, Alexei gets at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$. If it becomes the second largest piece, Boris gets at most $\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$.
Subcase 2c. Boris cuts $\frac{1}{5}$ into y and $\frac{1}{5}-y$, where $0 \le y \le \frac{1}{10}$.
Since $\frac{2}{5}-x \ge y$, the second smallest piece is at most $\frac{2}{5}-x$. Hence Boris gets at most $(\frac{2}{5}-x)+x=\frac{2}{5}$.
Subcase 2d. Boris cuts $\frac{2}{5}-x$.
Alexei gets at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.
We now show that Boris can be assured of getting $\frac{2}{5}$. We consider three cases.
Case 1. Alexei cuts 1 into $\frac{3}{5}-x$ and $\frac{2}{5}+x$, where $0 \le x \le \frac{1}{10}$.
Boris cuts $\frac{3}{5}-x$ into $\frac{2}{5}+x$ and $\frac{1}{5}-2x$. If Alexei cuts $\frac{1}{5}-2x$, Boris gets at least $\frac{2}{5}+x \ge \frac{2}{5}$. If Alexei cuts one $\frac{2}{5}+x$, Boris cuts the other $\frac{2}{5}+x$ in the same way and gets at least $\frac{2}{5}+x \ge \frac{2}{5}$.
Case 2. Alexei cuts 1 into $\frac{3}{5}+x$ and $\frac{2}{5}-x$, where $0 \le x \le \frac{1}{5}$.
Boris cuts $\frac{2}{5}-x$ into $\frac{1}{5}$ and $\frac{1}{5}-x$. If Alexei cuts either of these two pieces, Boris cuts $\frac{3}{5}-x$ into halves and gets at least $\frac{3}{10}+\frac{x}{2}+\frac{1}{2}(\frac{1}{5}-x)=\frac{2}{5}$. If Alexei cuts $\frac{3}{5}+x$ into y and $\frac{3}{5}+x-y$ where $0 \le y \le \frac{3}{10}+\frac{x}{2}$, Boris cuts the latter into $\frac{1}{5}+x$ and $\frac{2}{5}-y$. If $\frac{1}{5}-x \le y \le \frac{1}{5} \le \frac{2}{5}-y \le \frac{1}{5}+x$, Boris gets $y+(\frac{2}{5}-y)=\frac{2}{5}$. If $y \le \frac{1}{5}-x \le \frac{1}{5} \le \frac{1}{5}+x \le \frac{2}{5}-y$, Boris still gets $(\frac{1}{5}-x)+(\frac{1}{5}+x)=\frac{2}{5}$.
Case 3. Alexei cuts 1 into $\frac{4}{5}+x$ nd $\frac{1}{5}-x$, where $0 \le x \le \frac{1}{5}$.
Boris cuts $\frac{4}{5}+x$ into $\frac{3}{5}+x$ and $\frac{1}{5}$, and this is the same as Case 2.

7. Suppose we have an $a_1 \times a_2$ rectangle A and a $b_1 \times b_2$ rectangle B. Any rectangle $PQRS$ that can be constructed from copies of A has dimensions $(u_1a_1+u_2a_2)\times(v_1a_1+v_2a_2)$ for some non-negative integers $u_1,\ u_2, v_1$ and v_2. If $PQRS$ is similar to B, then

$$\frac{b_1}{b_2}=\frac{u_1a_1+u_2a_2}{v_1a_1+v_2a_2}.$$

We first consider the case where $\frac{a_1}{a_2}$ is rational, so that it is equal to $\frac{m_1}{m_2}$ for some positive integers m_1 and m_2. Then $\frac{b_1}{b_2}=\frac{u_1m_1+u_2m_2}{v_1m_1+v_2m_2}=\frac{n_1}{n_2}$ for some positive integers n_1 and n_2, so that it is also rational. Using n_1n_2 copies of B, we can construct a square of side $s=n_2b_1+n_1b_2$. Using m_1m_2 copies of this square, we can construct an $sm_1 \times sm_2$ rectangle which is similar to A.

We now consider the case where $\frac{a_1}{a_2}$ is irrational. We claim that in constructing the rectangle $PQRS$ with copies of A, all the copies must be in the same orientation. Let $PTUV$ be the largest subrectangle of $PQRS$ that can be constructed with copies of A all in the same orientation. Suppose U is in the interior of $PQRS$, as illustrated in the diagram below.



If the line TU can be extended without cutting in interior of a copy of A, then the space immediately below UV must be filled with copies of A in the same orientation as those above, as otherwise it contradicts the irrationality of $\frac{a_1}{a_2}$. However, now it contradicts the maximality of $PTUV$. Hence TU cannot be so extended, but this implies that VU can, and we have a contradiction as well. It follows that U must lie on QR or RS. We may assume by symmetry that it lies on QR, so that T coincides with Q. However, the space immediately below UV must be filled with copies of A in the same orientation as those above. This contradicts the maximality of $PTUV$ unless U coincides with R and V with S. Thus our claim is justified. Suppose this construction uses k_1k_2 copies of A in k_1 rows and k_2 columns for some positive integers k_1 and k_2. Then $\frac{k_1a_1}{k_2a_2} = \frac{b_1}{b_2}$ so that $\frac{k_2b_1}{k_1b_2} = \frac{a_1}{a_2}$. Hence we can construct a rectangle similar to A using k_1k_2 copies of B in k_2 rows and k_1 columns.