Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Spring 2003.**

1 Let S be an entire amount of money (\$2003),

a_i be amount of money in i-pocket, $i = 1, 2, \ldots, M$. Then

$$a_i < N, \qquad S = \sum_{i=1}^{M} a_i < MN. \tag{1}$$

Let us assume that each purse contains no less than M dollars in it. Let b_i be amount of money in i-purse. Then

$$b_i \geq M, \qquad S = \sum_{i=1}^{N} b_i \geq MN. \tag{2}$$

Contradiction.

2 Consider three cases:

(a) $n > 4$. Let us show that the first player has a winning strategy. On each of his subsequent moves, the first player colours a side which is one space away from one of already coloured sides. Note, that doing this, he creates a “store”, which he can use later; however, the second player can not, because of the nature of requirement. So, in the end of the game, after the first player’s move, we are left with cases:

(i) One uncoloured side is left (plus “store”). The second player has no move.

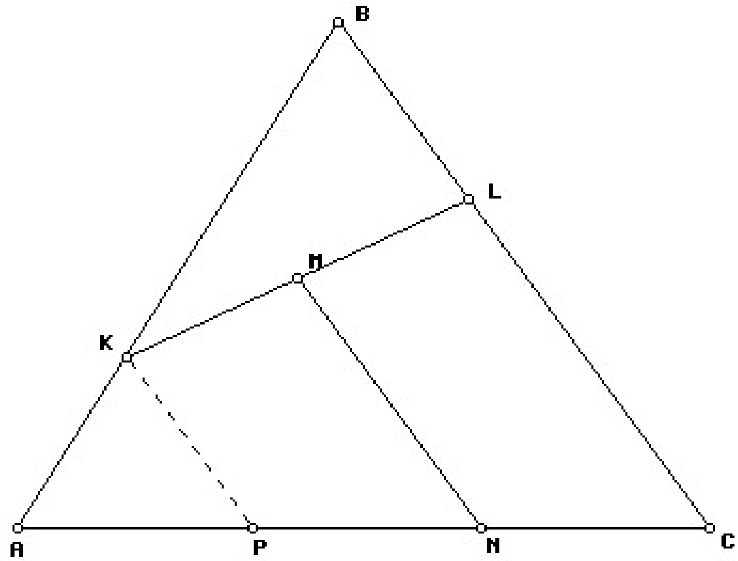
(ii) Two uncoloured sides are left (plus “store”). After the second player’s move, the first player wins.

(iii) Three uncoloured sides are left (plus “store”). After the second player’s move, the first player uses his ”store”, and wins on his next move.

(b) From above, we can see that the only chance for the second player to win is in the case (iii), when “store” is not yet created. It corresponds to the case $n = 4$. Really, the first player can not produce his second move and loses.

(c) $n = 3$. The first player wins.

3 Let us draw straight line $KP \parallel BC$ where P is a point on AC. Since $KLCP$ is a trapezoid, its midline $MN = \frac{1}{2}(KP + LC) = \frac{1}{2}(AK + LC) = \frac{1}{2}KL = KM = ML$. Then KL is a diameter of a circle passing through K, N, L and therefore $\angle KNL = 90°$.



4 We start from

Proposition. If a is an even number, then $5a \equiv 0 \pmod{10}$.

Proof is obvious.

Note, that in order to maintain the row of odd terms in a sequence, given the requirements, the last term's digit has to be odd and the term's largest digit even. Further, each addition could change the term's largest digit by at most 1. When it happens, the term's largest digit becomes odd and on next term the row of odd terms in a sequence is terminated. Since the term's largest digit stays the same through the row, the maximal number of terms cannot exceed five due to proposition. It is possible to have a row of five: 807, 815, 823, 831, 839.

5 The answer is negative.

Let us colour the board with black and white strips, black in excess. Note, that since dominoes placed horizontally and 1×3 rectangles placed vertically, each domino covers one black and one white square, meanwhile each rectangle covers three squares of the same colour.

Let us assume, that it is possible to tile 2003×2003 board by dominoes and rectangles. Let n be a number of dominoes. Then the numbers of black and white rectangles are equal to $(2003 \times 1002 - a)/3$ and $(2003 \times 1001 - a)/3$ respectively. Therefore, the difference between black and white rectangles is 2003 and has to be a multiple of 3. Contradiction.