International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper [1] **Fall 2003.**

1. Smallville is populated by unmarried men and women, some of them are mutual acquaintants. The City's two Official Matchmakers are aware of all the mutual acquaintances. One of them claimed: "I can arrange it so that every brown haired man will marry a woman with whom he is mutually acquainted." The other claimed, "I can arrange it so that every blond haired woman will marry a man with whom she is mutually acquainted." An amateur mathematician overheard their conversation and said, "Then both arrangements can be made at the same time!" Is he right?

2. Prove that one can represent every positive integer in the form $3^{u_1} \cdot 2^{v_1} + 3^{u_2} \cdot 2^{v_2} + \ldots + 3^{u_k} \cdot 2^{v_k}$ where $u_1 > u_2 > \cdots > u_k \geq 0$ and $0 \leq v_1 < v_2 < \cdots < v_k$ are integers.

3. An ant crawls on the outer surface of a rectangular box. The distance between two points on a surface is defined as the length of the shortest path the ant needs to crawl to reach one point from the other. Is it true that if the ant is at a vertex, then the opposite vertex is the point on the surface which is at the greatest distance away?

4. Triangle ABC has orthocentre H, incentre I and circumcentre O. K is the point where the incircle touches BC. If IO is parallel to BC, prove that AO is parallel to HK.

5. In a game, Boris has 1000 cards numbered $2, 4, \ldots, 2000$ while Anna has 1001 cards numbered $1, 3, \ldots, 2001$. The game lasts 1000 rounds. In an odd-numbered round, Boris plays any card of his. Anna sees it and plays a card of hers. The player whose card has the larger number wins the round, and both cards are discarded. An even-numbered round is played in the same manner except that Anna plays first. At the end of the game, Anna discards her unused card. What is the maximal number of rounds each player can guarantee to win, regardless of how the opponent plays?

6. Let O be the incentre of a tetrahedron $ABCD$ in which the sum of areas of the faces ABC and ABD is equal to the sum of areas of the faces CDA and CDB. Prove that midpoints of BC, AD, AC and BD lie on a plane passing through O.

7. Each cell of an $m \times n$ table is filled with a $+$ sign or a $-$ sign. Such a table is said to be *irreducible* if one cannot change all $-$ signs to $+$ signs by applying, as many times as desired, some permissible operation.

   (a) Suppose the permissible operation is to change the signs of all cells in a row or a column to the opposite signs. Prove that an irreducible table contains an irreducible 2×2 sub-table.

   (b) Suppose the permissible operation is to change the signs of all cells in a row, a column or a diagonal (which may be of any length, including those of length 1, consisting of a corner cell). Prove that an irreducible table contains an irreducible 4×4 sub-table.

Note: The problems are worth 4, 4, 6, 7, 7, 7 and 3+6 points respectively.

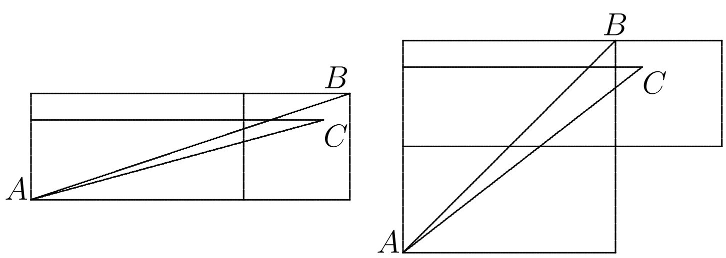
[1] Courtesy of Andy Liu

Solution to Senior A-Level Fall 2003

1. Construct a graph as follows. One group of vertices $X_1, X_2, \ldots, X_n$ represent the brown haired men. Another group of vertices $Y_1, Y_2, \ldots, Y_m$ represent the blond haired women. From each of these vertices, draw an arc pointing to the vertex representing the mate promised by the appropriate matchmaker. If a man represented by X_k is promised a woman who is not blond haired, a new vertex W_k is introduced to represent the mate. Similarly, if a woman represented by Y_k is promised a man who is not brown haired, a new vertex Z_k is introduced to represent the mate. Now some of these arcs may form a cycle. All vertices on such a cycle are X-vertices or Y-vertices, as W-vertices and Z-vertices have no out-going arcs. Moreover, since the arcs point alternately at the two groups, the cycle must be of even length. Hence the amateur mathematician can prescribe marriages, according to alternate arcs along the cycle, between brown haired men and blond haired women. There are also arcs which form a path. Such a path must consist of X-vertices and Y-vertices, ecept that it terminates at a W-vertex or an Z-vertex. The amateur mathematician can prescribe marriages according to alternate arcs along the path, starting with the initial arc. If the path is of even length, all marriages here are between brown haired men and blond haired women. If the path is of odd length, this is still the case except for the marriage corresponding to the terminating arc. In any cases, all brown haired men and all blond haired women are married off with mutual acquaintants.

2. We use induction, the basis being $1 = 3^0 2^0$. For any even integer $n \geq 2$, $\frac{n}{2}$ has a desirable expression $3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \cdots + 3^{u_k}2^{v_k}$. Hence $n = 3^{u_1}2^{v_1+1} + 3^{u_2}2^{v_2+1} + \cdots + 3^{u_k}2^{v_k+1}$ is also a desirable expression. For any odd integer $n \geq 3$, let t be the unique integer such that $3^{t+1} > n \geq 3^t$. If $n = 3^t$, then it has a desirable expression $3^t 2^0$. Otherwise, $n - 3^t$ has a desirable expression $3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \cdots + 3^{u_k}2^{v_k}$. Then $n = 3^t 2^0 + 3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \cdots + 3^{u_k}2^{v_k}$. Since $n - 3^t$ is even, $v_1 > 0$. Moreover, $3^{t+1} > n \geq 3^t + 3^{u_1}2^{v_1} \geq 2 \cdot 3^{u_1}$. Hence $2 \cdot 3^t > 2 \cdot 3^{u_1}$ so that $t > u_1$. It follows that our expression for n is a desirable one.

3. Let A and B be opposite vertices of a $4 \times 4 \times 8$ box and let C be a point on the 4×4 face with B as a vertex, such that the distance from C to each side containing B is 1. To go from A to B or C efficiently, we unfold the box in one of two ways, both shown in the diagram below, and travel in a straight line. In the diagram on the left, $AB = \sqrt{12^2 + 4^2} = \sqrt{160}$ while $AC = \sqrt{11^2 + 3^2} = \sqrt{130}$. In the diagram on the right, $AB = \sqrt{8^2 + 8^2} = \sqrt{128}$ while $AC = \sqrt{9^2 + 7^2} = \sqrt{130}$. All other ways of unfolding the box yield longer distances for AB and AC. Hence the minimum distance from A to B is $\sqrt{128}$ and that from A to C is $\sqrt{130}$, showing that C is further from A than B.



4. We first establish two well-known results.

Lemma 1.

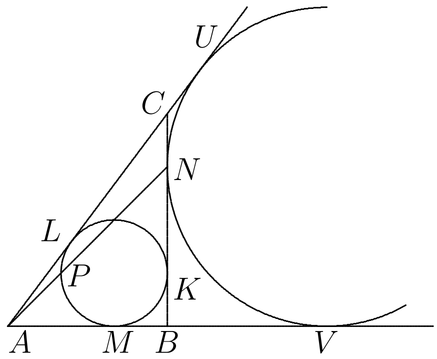
Let O and H be the circumcentre and the orthocentre of triangle ABC respectively, and let D be the midpoint of BC. Then $AH = 2DO$.

Proof:

Let G be the centroid of triangle ABC. Perform a central projection about G carrying A into D. This carries B into the midpoint E of CA and C into the midpoint F of AB. Hence it carries H into the orthocentre of triangle DEF, which is O. It follows that O, G and H lie on a line (called the *Euler line* of ABC). Since OD and AH are both perpendicular to BC, they are parallel. Hence triangles DOG and AHG are similar. Since $AG = 2GD$, we have $AH = 2DO$.

Lemma 2.

PK is a diameter of the incircle of triangle ABC where K is its point of tangency with BC. The extension of AP cuts BC at N. Then $BN = CK$.



Proof:

Perform a central projection about A carrying P into N. This carries the incircle of ABC into its excircle opposite A, N being the point of tangency with BC. Let the incircle touch CA at L and AB at M, and let the excircle touch CA at U and AB at V. Then

$$\begin{aligned} 2BN &= BN + BK - NK \\ &= BU + BM - NK \\ &= AV - AM - NK \\ &= AU - AL - NK \\ &= CV + CL - NK \\ &= CN + CK - NK \\ &= 2CK. \end{aligned}$$

Dividing by 2 yields the desired result.

We now return to the original problem. Let D and N be as in the Lemmas. That OI is parallel to BC means that $DO = IK$. By Lemma 1, $AH = 2DO = 2IK = PK$. Since AH is parallel to PK, $AHKP$ is a parallelogram so that AP is parallel to HK. By Lemma 2, $ND = BD - BN = CD - CK = DK$. Let the line through D perpendicular to BC cut AN at Q. Then $DQ = \frac{1}{2}PK = DO$. Since O also lies on this line and on the same side of BC as Q, Q and O coincide. Hence O lies on AN, and AO is parallel to HK.

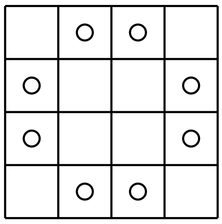
5. Let us first play the game with 5 cards, with Boris holding 2 and 4 and Anna holding 1, 3 and 5. If Boris leads 2, Anna wins both rounds by playing 3 and then leading 5. If Boris leads 4, Anna wins both rounds by playing 5 and then leading 3. We shall prove by induction on n that if there are $4n+1$ cards, than Boris can guarantee winning $n-1$ rounds while Anna can guarantee winning $n+1$ rounds. The best strategy for Boris is to lead 2 in the first round. If Anna is going to win this round, she should definitely play 3. Moreover, there is no reason why she should lose this round by playing 1, because 1 and 3 are equivalent after 2 has been discarded. Hence we may assume that Anna wins the first round with 3. If in the second round Anna does not lead $4n+1$, Boris can win that round by playing the lowest card above Anna's. At this point, although there are now some gaps among the $4n-3$ cards still in play, the holdings of Boris and Anna are in the alternating pattern. Hence we may play the balance of the game as though the numbers on the cards have been adjusted to run from 1 to $4n-3$, and reach the desired conclusion by induction. Suppose Anna leads $4n+1$ in the second round. Clearly, Boris should concede by playing 4, but he can win the next round by leading $4n$, forcing Anna to concded with 1 or 5 (now equivalent) and restoring the alternating pattern. If in any subsequent round, Anna does not lead her highest card, Boris can win that round and at the same time restore the alternating pattern. If Anna continues to lead her highest card, Boris can do likewise. He will then lose the first and the last rounds, and wins every second round in between, winning altogether $n-1$ rounds. Anna's simplest strategy is to win the first round by playing the lowest card above Boris's, and leading 1 in the second round. The desired conclusion then follows from the induction hypothesis.

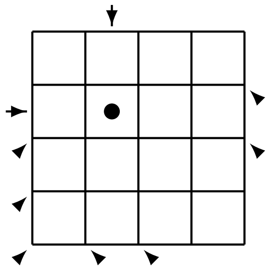
6. Denote by $[P]$ the area of the polygon P. We are given that $[ABC]+[ABD]=[ACD]+[BCD]$. Denote the common value by S. Let V be the volume of $ABCD$ and r be its inradius. Then $V=\frac{r}{3}([ABC]+[ABD]+[ACD]+[BCD])$ so that $r=\frac{3V}{2S}$. Let $E,\ F,\ G$ and H be the respective midpoints of $BC,\ AC,\ AD$ and BD. Let $a,\ b,\ c$ and d be the altitudes of $ACD,\ BCD, ABC$ and ABD from $A,\ B,\ C$ and D respectively. Then $S=[ABC]+[ABD]=\frac{1}{2}AB(c+d)$ so that $2S=AB(c+d)$. Similarly, $2S=CD(a+b)$. Since EF and GH are both parallel to AB while EH and FG are both parallel to CD, $EFGH$ is a parallelogram. Draw a line between EF and GH and parallel to both, such that its distances from EF and GH are in the ratio $c:d$. Draw another line between EH and FG and parallel to both, such that its distance from EH and FG are in the ratio $b:a$. Now the altitude of $ABCD$ from D is equal to $\frac{3V}{[ABC]}$. Hence the distance from G or H to ABC is $\frac{3V}{cAB}$ and the distance from I to ABC is $\frac{3V}{cAB}\cdot\frac{c}{c+d}=\frac{3V}{2S}$. Similarly, the distances from I to $ABD,\ AACD$ and BCD are all equal to $\frac{3V}{2S}$. It follows that I coincides with O, so that O is indeed coplanar with $E,\ F,\ G$ and H.

7. (a) We first note that a 2×2 table is irreducible if and only if it contains an odd number of $-$ signs. It is easy to verify that if the number of $-$ signs is even, all of them can be changed to $+$ signs by the permissible operation. However, since this operation cannot change the parity of the number of $-$ signs, a 2×2 table with an odd number of $-$ signs is indeed irreducible. If an $m\times n$ table contains an irreducible 2×2 sub-table, it is clearly irreducible since we cannot even change all $-$ signs in the sub-table into $+$ signs. Suppose there are no irreducible sub-tables. By applying the permissible operation to columns headed by a $-$ sign, we can make all signs in the first row $+$ signs. We claim that the second row consists only of $-$ signs or only of $+$ signs. Otherwise, there will

be a $-$ sign next to a $+$ sign. Together with the $+$ signs in the first row, these four will constitute an irreducible 2×2 sub-table. By applying the permissible operation to the second row if necessary, we can make it a row of $+$ signs. In the same way, we can fix the remaning rows, showing that a table without an irreducible sub-table is reducible.

(b) We claim that a 4×4 table is irreducible if and only if it contains an odd number of $-$ signs in its eight edge squares, which are marked with circles in the diagram on the left. Since the permissible operation always affects an even number of these squares, it cannot change the parity of the number of $-$ signs in them. It follows that such a 4×4 table is indeed irreducible. If an $m \times n$ table contains an irreducible 4×4 sub-table, it is clearly irreducible since we cannot even change all $-$ signs in the edge squares of the sub-table into $+$ signs.





Suppose a 4×4 table has an even number of $+$ signs in its eight edge squares. By applying the permissible operation one column at a time if necessary, we can change the four signs in the edge squares on the top two rows into $+$ signs. If the two signs in the edge squares on the third row are the same, we can change both into $+$ signs by applying the permissible operation along the that row. Then the two signs in the edge squares on the bottom row must be the same, and can be changed into $+$ signs. If the two signs in the edge squares on the third row are different, then the two signs in the edge squares on the bottom row must also be different. They can be changed into $+$ signs by applying the permissible operation along the short diagonals involving only these squares, preceded by an application along the third row if necessary. The diagram on the right shows how one can apply the permissible operation nine times along the row, columnn and diagonals as indicated, so that a $-$ sign in the central square marked with a black circle can be changed into a $+$ sign without affecting any other squares. Finally, the corner squares can be fixed individually as diagonals of length 1. This justifies our earlier claim that a 4×4 table i irreducible if and only if the number of $-$ signs in its eight edge squares is even. Suppose there are no irreducible sub-tables in an $m \times n$ table. We first change all signs in the top three rows into $+$ signs as follows. We proceed column by column from left to right. By applying the permissible operation along the column if necessary, we can make the sign in the second row a $+$ sign. By applying the permissible operation along a down-diagonal passing through the square in the top row and along an up-diagonal passing through the square in the third row, we can make those $+$ signs too. In the fourth row, apart from the two end squares, the signs in all the others must be the same, as otherwise we would have a irreducible 4×4 sub-table. By applying the permissible operation along this row if necessary., we can make them all $+$ signs. The end squares can be dealt with by applying the permissible operation in diagonals which do not affect any other squares in the top four rows. In the same way, we can fix the table one row at a time and change all the signs into $+$ signs.