International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Fall 2003.**

1. An increasing arithmetic progression consists of one hundred positive integers. Is it possible that every two of them are relatively prime?

2. Smallville is populated by unmarried men and women, some of them are mutual acquaintants. The City's two Official Matchmakers are aware of all the mutual acquaintances. One of them claimed: "I can arrange it so that every brown haired man will marry a woman with whom he is mutually acquainted." The other claimed, "I can arrange it so that every blond haired woman will marry a man with whom she is mutually acquainted." An amateur mathematician overheard their conversation and said, "Then both arrangements can be made at the same time!" Is he right?

3. Determine all positive integers k such that there exist positive integers m and n satisfying $m(m+k) = n(n+1)$.

4. In chess, a bishop attacks any square on the two diagonals that contain the square on which it stands, including that square itself. Several squares on a 15×15 chessboard are to be marked so that a bishop placed on any square of the board attacks at least two of the marked squares. Determine the minimal number of such marked squares.

5. Prove that $135^\circ \le \angle OAB + \angle OBC + \angle OCD + \angle ODA \le 225^\circ$ for any point O inside a square $ABCD$.

6. An ant crawls on the outer surface of a rectangular box. The distance between two points on a surface is defined as the length of the shortest path the ant needs to crawl to reach one point from the other. Is it true that if the ant is at a vertex, then the opposite vertex is the point on the surface which is at the greatest distance away?

7. In a game, Boris has 1000 cards numbered $2, 4, \ldots, 2000$ while Anna has 1001 cards numbered $1, 3, \ldots, 2001$. The game lasts 1000 rounds. In an odd-numbered round, Boris plays any card of his. Anna sees it and plays a card of hers. The player whose card has the larger number wins the round, and both cards are discarded. An even-numbered round is played in the same manner except that Anna plays first. At the end of the game, Anna discards her unused card. What is the maximal number of rounds each player can guarantee to win, regardless of how the opponent plays?

Note: The problems are worth 4, 5, 5, 6, 7, 7 and 8 points respectively.

[1] Courtesy of Andy Liu.

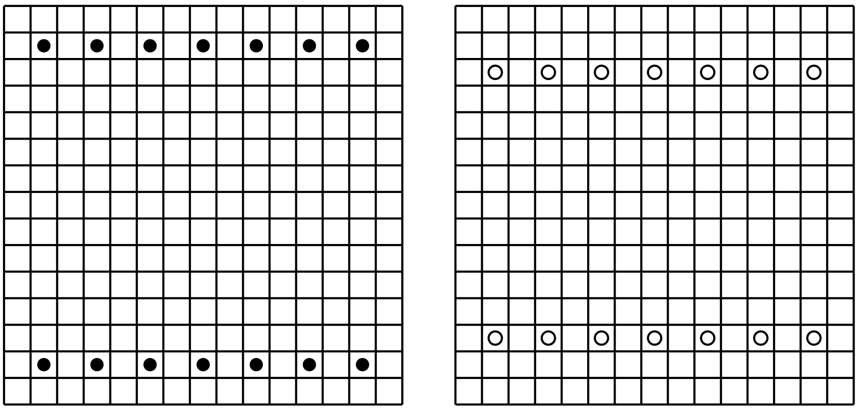
Solution to Senior A-Level Fall 2003

1. Let the arithmetic progression be $\{a_k\}$ where $a_k = k(101!) + 1$ for $k = 1, 2, \ldots, 100$. Let d be the greatest common divisor of a_i and a_j where $i < j$. Then d divided $a_j - a_i = (j-i)(101!)$. If d has a prime divisor p, then $p \leq 101$. On the other hand, since p divides $i(100!) + 1$, we must have $p > 101$. This contradiction shows that d has no prime divisors. It follows that $d = 1$.

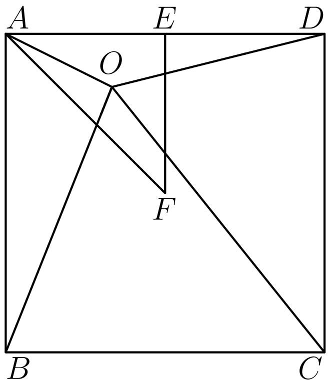
2. Construct a graph as follows. One group of vertices $X_1, X_2, \ldots, X_n$ represent the brown haired men. Another group of vertices $Y_1, Y_2, \ldots, Y_m$ represent the blond haired women. From each of these vertices, draw an arc pointing to the vertex representing the mate promised by the appropriate matchmaker. If a man represented by X_k is promised a woman who is not blond haired, a new vertex W_k is introduced to represent the mate. Similarly, if a woman represented by Y_k is promised a man who is not brown haired, a new vertex Z_k is introduced to represent the mate. Now some of these arcs may form a cycle. All vertices on such a cycle are X-vertices or Y-vertices, as W-vertices and Z-vertices have no out-going arcs. Moreover, since the arcs point alternately at the two groups, the cycle must be of even length. Hence the amateur mathematician can prescribe marriages, according to alternate arcs along the cycle, between brown haired men and blond haired women. There are also arcs which form a path. Such a path must consist of X-vertices and Y-vertices, ecept that it terminates at a W-vertex or an Z-vertex. The amateur mathematician can prescribe marriages according to alternate arcs along the path, starting with the initial arc. If the path is of even length, all marriages here are between brown haired men and blond haired women. If the path is of odd length, this is still the case except for the marriage corresponding to the terminating arc. In any cases, all brown haired men and all blond haired women are married off with mutual acquaintants.

3. Multiply the given equation by 4 and rearranging terms, we have $(2m+k)^2 - (2n+1)^2 = k^2 - 1$. Suppose k is odd. We may set $2m + k + 2n + 1 = \frac{k^2-1}{2}$ and $2m + k - 2n - 1 = 2$, yielding $m = \frac{(k-2)^2-1}{8}$ and $n = \frac{k^2-1}{8} - 1$. Both are integers since $x^2 \equiv 1 \pmod 8$ for odd x, and both are positive if $k \geq 5$. For $k = 3$, we have $(2m+3)^2 - (2n+1)^2 = 8$, but the only two squares differing by 8 are 9 and 1. Hence we must have $m = n = 0$. Suppose k is even. e may set $2m+k+2n+1 = k^2-1$ and $2m+k-2n-1 = 1$, yielding $m = \frac{k^2}{4} - \frac{k}{2}$ and $n = \frac{k^2}{4} - 1$. Both are clearly integers, but are only positive if $k \geq 4$. For $k = 2$, we have $(2m+2)^2 - (2n+1)^2 = 3$, but the only two squares differing by 3 are 4 and 1. Again we must have $m = n = 0$. In summary, positive integers m and n exist if and only if $k \geq 4$.

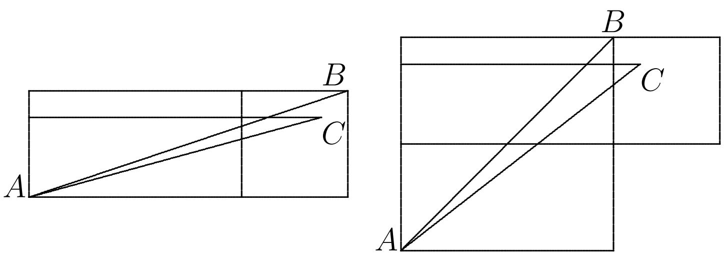
4. Colour the squares in the 15×15 board in the usual chessboard patterm, with black squares at the corners. In the diagram on the left, 14 black squares are marked with black circles. It is easy to verify that any bishop placed on a black square will attack at least 2 of the marked black squares. To show that 14 is optimal, consider the 28 black squares along the edge of the board. Each must attack at least 2 of the marked black squares. Yet any black square can be attacked by at most 2 of the black squares along the edge. Similarly, 14 white squares can be marked as shown in the diagram on the right, with white circles. As with black sqaures, this is both necessary and sufficient.



5. Let E be the midpoint of AD and let F be the centre of $ABCD$. We may assume by symmetry that O is in triangle AEF. Hence $OA \le OD$ and $OC \ge OB$, so that $\angle ODA \le \angle OAD$ and $\angle OBC \ge \angle OCB$. Now $90^\circ = \angle OAB + \angle OAD \ge \angle OAB + \angle ODA \ge \angle OAB \ge 45^\circ$ while $90^\circ + 45^\circ \ge \angle OBC + \angle OCD \ge \angle OCB + \angle OCD = 90^\circ$. Addition yields the desired result.



6. Let A and B be opposite vertices of a $4 \times 4 \times 8$ box and let C be a point on the 4×4 face with B as a vertex, such that the distance from C to each side containing B is 1. To go from A to B or C efficiently, we unfold the box in one of two ways, both shown in the diagram below, and travel in a straight line. In the diagram on the left, $AB = \sqrt{12^2 + 4^2} = \sqrt{160}$ while $AC = \sqrt{11^2 + 3^2} = \sqrt{130}$. In the diagram on the right, $AB = \sqrt{8^2 + 8^2} = \sqrt{128}$ while $AC = \sqrt{9^2 + 7^2} = \sqrt{130}$. All other ways of unfolding the box yield longer distances for AB and AC. Hence the minimum distance from A to B is $\sqrt{128}$ and that from A to C is $\sqrt{130}$, showing that C is further from A than B.



7. Let us first play the game with 5 cards, with Boris holding 2 and 4 and Anna holding 1, 3 and 5. If Boris leads 2, Anna wins both rounds by playing 3 and then leading 5. If Boris leads 4, Anna wins both rounds by playing 5 and then leading 3. We shall prove by induction on n that if there are $4n+1$ cards, than Boris can guarantee winning $n-1$ rounds while Anna can guarantee winning $n+1$ rounds. The best strategy for Boris is to lead 2 in the first round. If Anna is going to win this round, she should definitely play 3. Moreover, there is no reason why she should lose this round by playing 1, because 1 and 3 are equivalent after 2 has been discarded. Hence we may assume that Anna wins the first round with 3. If in the second round Anna does not lead $4n+1$, Boris can win that round by playing the lowest card above Anna's. At this point, although there are now some gaps among the $4n-3$ cards still in play, the holdings of Boris and Anna are in the alternating pattern. Hence we may play the balance of the game as though the numbers on the cards have been adjusted to run from 1 to $4n-3$, and reach the desired conclusion by induction. Suppose Anna leads $4n+1$ in the second round. Clearly, Boris should concede by playing 4, but he can win the next round by leading $4n$, forcing Anna to concded with 1 or 5 (now equivalent) and restoring the alternating pattern. If in any subsequent round, Anna does not lead her highest card, Boris can win that round and at the same time restore the alternating pattern. If Anna continues to lead her highest card, Boris can do likewise. He will then lose the first and the last rounds, and wins every second round in between, winning altogether $n-1$ rounds. Anna's simplest strategy is to win the first round by playing the lowest card above Boris's, and leading 1 in the second round. The desired conclusion then follows from the induction hypothesis.