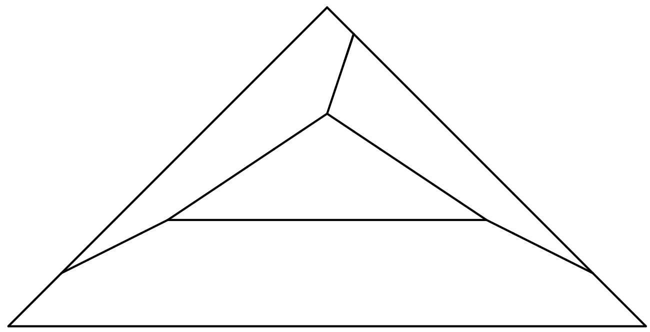
Solution to Junior O-Level Spring 2002

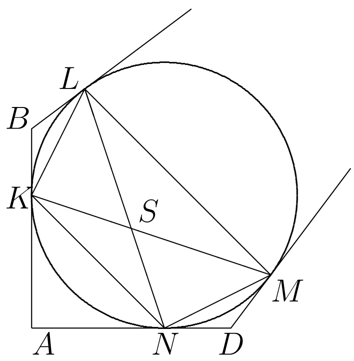
1. The only common divisors of $49 \times 51 = 3 \times 7^2 \times 17$ and $99 \times 101 = 3^2 \times 11 \times 101$ are 1 and 3. Since $a < b$, $ab > 1$. So $ab = 3$ and we must have $a = 1$ and $b = 3$.

2. If either x or y is odd, $x^2 + xy + y^2$ is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, $x^2 + xy + y^2$ is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6. It follows that the only possibility is for both x and y to be multiples of 10, so that $x^2 + xy + y^2$ will indeed be a multiple of 100.

3. One such dissection is shown in the diagram below.



4. Since BK and BL are tangents, $\angle BKL = \angle KML = \angle BLK$. Denote their common value by θ. Then $\angle BKL = 180° - 2\theta$. Similarly, $\angle DMN = \angle MLN = \angle DNM$. Denote their common value by ϕ. Then $\angle MDN = 180° - 2\phi$. Now $\angle KSL = \angle SLM + \angle SML = \theta + \phi$. Similarly, $\angle MSN = \theta + \phi$. Since $SKBL$ is cyclic, $\angle KBL + \angle KSL = 180°$, which implies that $\theta = \phi$. Then $\angle MDN + \angle MSN = 180°$, which implies that $SMDN$ is cyclic.



5. (a) Weigh 64 of the coins against the other 64. If they balance, discard one set. Weigh 32 of the remaining ones against the other 32, and continue. If they always balance, then after 6 weighings, we are down to 2 coins which must consist of a heavy one and a light one. Suppose balance is not achieved somewhere along the way. We may as well assume that it occurs at the first weighing. In the second weighing, weigh 32 coins from the heavier side against 32 coins from the lighter side. If they balance, discard this 64 coins. If not, discard the 64 coins not involved in the second weighing. Continuing this way, we will be down to 2 coins after 7 weighings. They must consist of a heavy one and a light one.

   (b) Weigh 4 of the coins against the other 4. If they balance, discard one set. Weigh 2 of the remaining 4 coins against the other 2. If they balance, take both coins from one side. If not, take 1 coin from each side. Suppose one side is heavier in the first weighing. Weigh 2 of these coins against the other 2. If they balance, all 4 are heavy. Take 1 of them and 1 from the lighter side in the first weighing. If they do not balance, then the heavier side consists of 2 heavy coins while the lighter side consists of 1 heavy and 1 light coin. We can accomplish the task by taking the 2 coins on the lighter side.