Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2002.**

1 [**4**] Each of two children (John and Mary) selected a natural number and communicated it to Bill. Bill wrote down the sum of these numbers on one card and their product on another, hid one card and showed the other to John and Mary.

John looked at the number (which was 2002) and declared that he was not able to determine the number chosen by Mary. Knowing this, Mary said that she was also not able to determine the number chosen by John.

What was the number chosen by Mary?

2

a [**1**] A test was conducted in a class. It is known that at least $\frac{2}{3}$ of the problems were hard: each such problem was not solved by at least $\frac{2}{3}$ of the students. It is also known that at least 2/3 of students passed the test: each such student solved at least $\frac{2}{3}$ of the suggested problems.

Is this situation possible?

b [**1**] The same question with $\frac{2}{3}$ replaced by $\frac{3}{4}$.

c [**2**] The same question with $\frac{2}{3}$ replaced by $\frac{7}{10}$.

3 [**5**] Several straight lines such that no two of them are parallel, cut the plane into several regions. A point A is marked inside of one region. Prove that a point, separated from A by each of these lines, exists if and only if A belongs to unbounded region.

4 [**5**] Let x, y, z be any three numbers from the open interval $(0, \pi/2)$. Prove the inequality

$$\frac{x \cdot \cos x + y \cdot \cos y + z \cdot \cos z}{x + y + z} \leq \frac{\cos x + \cos y + \cos z}{3}.$$

5 [**5**] Each term of an infinite sequence of natural numbers is obtained from the previous term by adding to it one of its nonzero digits. Prove that this sequence contains an even number.

Keep the problem set.
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