SOLUTIONS OF TOURNAMENT OF TOWNS

Spring 2001, Level 0, Senior (grades 11-OAC)

Problem 1 [3] A bus that moves along a 100 km route is equipped with a computer, which predicts how much more time is needed to arrive at its final destination. This prediction is made on the assumption that the average speed of the bus in the remaining part of the route is the same as that in the part already covered. Forty minutes after the departure of the bus, the computer predicts that the remaining travelling time will be 1 hour. And this predicted time remains the same for the next 5 hours. Could this possibly occur? If so, how many kilometers did the bus cover when these 5 hours passed? (Average speed is the number of kilometers covered divided by the time it took to cover them.)

Solution. Let $S(t)$ be a distance covered by the bus for a time t. If the described situation is possible then for any moment $t \geq \frac{2}{3}$ (in hours) we have

$$\frac{100 - S(t)}{1} = \frac{S(t)}{t}$$

or

$$S(t) = \frac{100t}{1+t}. \qquad (*)$$

It is easy to see that $S(t)$ is a continuous monotone increasing function on $(0, \infty)$; this means that the bus is moving toward its destination. Moreover the distance expressed by (*) means that at any moment t the estimated remaining time will be 1 hour. Substituting $t = 5\frac{2}{3}$ into $(*)$ we get that $S(t) = 85$ km.

Problem 2 [4] The decimal expression of the natural number a consists of n digits, while that of a^3 consists of m digits. Can $n + m$ be equal to 2001?

Solution. The fact that the decimal expression of a natural number a consists of n digits means that

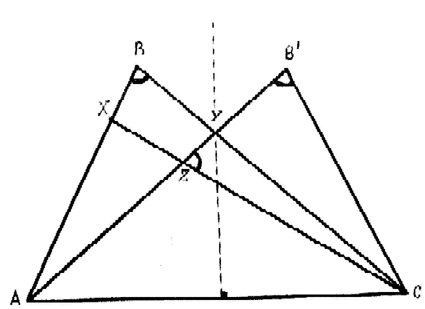
$$10^{n-1} \leq a < 10^n;$$

thus

$$10^{3n-3} \leq a^3 < 10^{3n}.$$

So, $m \in \{3n-2, 3n-1, 3n\}$ and $n + m \in \{4n-2, 4n-1, 4n\}$. Therefore $n + m \neq 1 \pmod 4$ and the answer is negative.

Problem 3 [4] Points X and Y are chosen on the sides AB and BC of the triangle $\triangle ABC$. The segments AY and CX intersect at the point Z. Given that $AY = YC$ and $AB = ZC$ prove that the points B, X, Z, and Y lie on the same circle.



Solution. Let us construct B' symmetrical to B with respect to the straight line passing through Y perpendicular to AC. We get that $\triangle ABY$ is congruent to $\triangle YB'C$; so $\angle ABC = \angle AB'C$. Since $ZC = AB = B'C$ we have $\angle AB'C = \angle B'CZ$. This implies that $\angle XZY = 180° - \angle XBY$ which means that the points B, X, Z, Y lie on the same circle.

Problem 4 [5] Two persons play a game on a board divided into 3×100 squares. They move in turn: the first places tiles of size 1×2 lengthwise (along the long axis of the board), the second, in the perpendicular direction. The loser is the one who cannot make a move. Which of the players can always win (no matter how his opponent plays), and what is the winning strategy?

Solution. Let us partition the board into 25 parts 3×4. The first player's strategy is to put tiles into the middle lines of these parts. For his first move he chooses any part; if the second player puts his tile into the same part then the first player chooses any free part for his next move; otherwise he puts his tile in the same part that the second player did. This guarantees at least 25 moves for the first player, leaving not more than 25 additional moves for the second player. However, the first player is guaranteed at least 25×2 the other moves (above and below his tiles) and the second player can not prevent him from making those moves.

Problem 5 [5] Nine points are drawn on the surface of a regular tetrahedron with an edge of 1 cm. Prove that among these points there are two located at a distance (in space) no greater than 0.5 cm.

Solution. Let us partition a tetrahedron surface into 16 congruent triangles, dividing each face by its middle lines. Now let us create 8 regions by painting these triangles according to the following rule: the triangles related to the same tetrahedron vertex we paint with one colour; so we use four different colours for 12 such triangles and another four for the rest of them. According to the Pigeonhole principle, at least two points belong to the same region. This only leaves us to prove that if two points belong to the same region (both types) then the distance between them cannot exceed 0.5.