International Mathematics

22nd Tournament of Towns

Spring 2001, Advanced Level

Solutions

JUNIOR (GRADES 7, 8, 9 AND 10)

1. [3] In a certain country 10% of the employees get 90% of the total salary paid in this country. Supposing that the country is divided in several regions, is it possible that in every region the total salary of any 10% of the employees is no greater than 11% of the total salary paid in this region?

Solution. Yes, it is possible. Assume there are 100 employees and 2 regions A and B in the country. Assume also that there are 10 people in region A and 90 people in region B. Let the salary of each employee in region A be \$81,000 and the salary of each employee in region B be \$1,000.

The salary of 10 people (which is 10% of the employees) in region A is \$810,000 (which is 90% of the total salary). Also the salary of any 10% of employees in region A (i.e. of any person) is 10% of the salary paid in this region. Clearly, the same holds for region B.

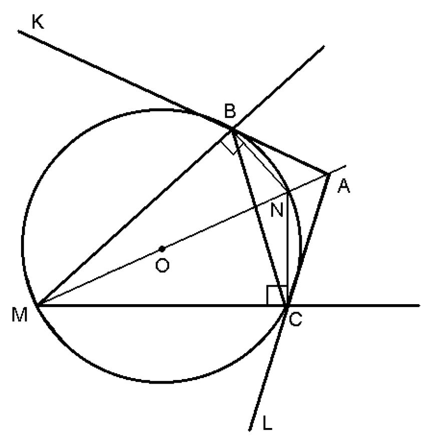
2. [5] In three piles there are 51, 49, and 5 stones, respectively. You can combine any two piles into one pile or divide a pile consisting of an even number of stones into two equal piles. Is it possible to get 105 piles with one stone in each?

Solution. No, it is not. Note that if at one step the number of stones in each pile is divisible by an odd integer, then at the next step the number of stones in each pile is divisible by the same integer. Clearly, at the very first step we only can obtain either two piles of 100 and 5 stones, or two piles of 56 and 49 stones, or two piles of 54 and 51 stones. In each case the number of stones in two piles has an odd divisor (5, 7, and 3, respectively) greater than 1. Thus, we cannot obtain 105 piles of 1 stone each, since the common divisor in that case is 1.

3. [5] Point A lies inside an angle with vertex M. A ray issuing from point A is reflected in one side of the angle at point B, then in the other side at point C and then returns back to point A (the ordinary rule of reflection holds). Prove that the center of the circle circumscribed about triangle $\triangle BCM$ lies on line AM.

Solution. Let N be the intersection point of the lines through B and C orthogonal to MB and MC respectively. In quadrilateral $MBNC$ angles B and C are right angles, thus point N lies on the circle circumscribed about $\triangle BCM$ and MN is the diameter of this circle.

On the other hand, because of the reflection law at points B and C, lines BN and CN are bisectors in triangle $\triangle ABC$. Thus, we only need to show that point M also lies on the bisector AN. Indeed, BM and CM are bisectors of angles $\angle CBK$ and $\angle BCL$, respectively (again by the reflection at points B and C). It follows that the distance from M to lines AK and AL is the same. Therefore, point M belongs to the bisector of angle A.



4. [5] Several non-intersecting diagonals divide a convex polygon into triangles. At each vertex of the polygon the number of triangles adjacent to it is written. Is it possible to reconstruct all the diagonals using these numbers if the diagonals are erased?

Solution. Yes, it is possible. First, we will prove that there always exists a vertex with number 1 written next to it. Indeed, each diagonal splits the polygon into two polygons. Let us choose a diagonal for which one of the polygons has minimal number of vertices (the other polygon will have maximal number of vertices). Clearly, there are no other diagonals inside the "smallest" polygon (otherwise it would not be "smallest"), therefore it is a triangle. Denote it $\triangle ABC$, where BC is the chosen diagonal. The number corresponding to vertex A is 1.

Now take any vertex (call it A) with number 1 next to it. Then the diagonal connecting its two adjacent vertices B and C is one of the erased diagonals. Let us draw it and consider new polygon where vertex A is omitted. Let us also subtract 1 from the numbers at vertices B and C (since the number of triangles in our new polygon adjacent to these vertices is smaller by 1). Our new polygon has fewer number of vertices and satisfies the conditions of the problem, so we can repeat our procedure until we end up with a triangle. All the diagonals are now reconstructed.

5. (a) [3] One black and one white pawn are placed on a chessboard. You may move the pawns in turn to the neighbouring empty squares of the chessboard using vertical and horizontal moves. Can you arrange the moves so that every possible position of the two pawns will appear on the chessboard exactly once?

(b) [4] Same question, but you don't have to move the pawns in turn.

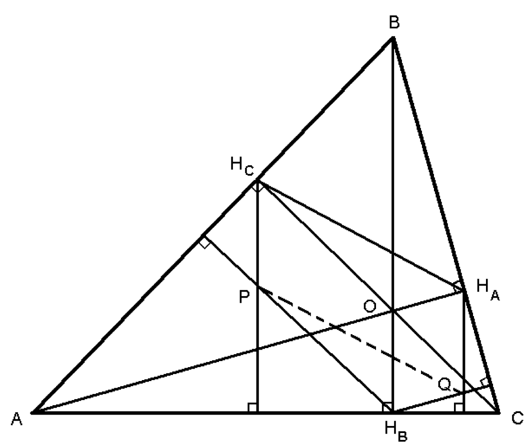
Solution. (a) Assume we can arrange the moves so that every possible position appears exactly once. Consider any square which is empty at the beginning and at the end of our moves. Then there are exactly 63 positions (we call them "special") when the white pawn is placed on this square and the black one is placed on some other square. On the other hand, all special position split into pairs: when the white pawn moves to this square (this is a special position) and after the next move of the black pawn (this is again a special position). All these pairs are different since we assumed that every possible position appears only once. But 63 is odd, which is a contradiction. The answer is no.

(b) Again the answer is no. Let us call a position "even" if the pawns are placed on the squares of the same colour, and "odd" otherwise. Note that even and odd positions alternate. Therefore the number of even and odd positions differs at most by 1. On the other hand, the number of even positions is $64 \cdot 31$ (indeed, we can put the white pawn at any of 64 squares and the black pawn at any of 31 squares of the same colour as the first one). Similarly, the number of odd positions is $64 \cdot 32$ (again we can put the white pawn at any square and the black pawn at any of 32 squares of the opposite colour to the first one). Therefore, the number of even and odd positions differs by 64, and thus we cannot arrange moves so that every possible position appears exactly once.

6. [7] Let AH_A, BH_B and CH_C be the altitudes of triangle $\triangle ABC$. Prove that the triangle whose vertices are the intersection points of the altitudes of triangles $\triangle AH_BH_C$, $\triangle BH_AH_C$ and $\triangle CH_AH_B$ is equal to triangle $\triangle H_AH_BH_C$.

Solution. Let O be the intersection point of the altitudes of triangle $\triangle ABC$. Let P, Q be the intersection points of the altitudes of triangles $\triangle AH_BH_C$ and $\triangle CH_AH_B$, respectively. We will prove that side PQ is equal to side H_CH_A. The proof of equality of the other two pairs of sides is the same.

We are going to show that H_CPQH_A form a parallelogram, and hence $PQ = H_CH_A$. First, note that H_CPH_BO is a parallelogram. Therefore, H_CP and OH_B are equal and parallel. On the other hand, H_AQH_BO is also a parallelogram. Therefore, OH_B and H_AQ are equal and parallel. We get H_CP and H_AQ are equal and parallel, so H_CPQH_A is a parallelogram.



7. Alex thinks of a two-digit integer (any integer between 10 and 99). Greg is trying to guess it. If the number Greg names is correct, or if one of its digits is equal to the corresponding digit of Alex's number and the other digit differs by one from the corresponding digit of Alex's number, then Alex says "hot"; otherwise, he says "cold". (For example, if Alex's number was 65, then by naming any of 64, 65, 66, 55 or 75 Greg will be answered "hot", otherwise he will be answered "cold".)

(a) [2] Prove that there is no strategy which guarantees that Greg will guess Alex's number in no more than 18 attempts.

(b) [3] Find a strategy for Greg to find out Alex's number (regardless of what the chosen number was) using no more than 24 attempts.

(c) [3] Is there a 22 attempt winning strategy for Greg?

Solution. (a) Assume that Greg has a strategy which guarantees that he will guess Alex's number in no more than 18 attempts. If Greg says $\overline{ab}$ and Alex says "cold" then the numbers $\overline{a(b-1)}$, $\overline{a(b+1)}$, $\overline{(a-1)b}$, $\overline{(a+1)b}$ and $\overline{ab}$ cannot be the Alex's number. Therefore, no matter what strategy Greg has there is a situation when the first 17 answers were "cold". Indeed, on each step Greg can "cover" at most 5 numbers, so there will be $90 - 17 \cdot 5 = 5$ numbers which will not be covered and one of them could have been Alex's. But even if the 18th answer is "hot", still there are five numbers that could have been Alex's number and Greg will not be able to tell which one it was.

(b), (c) We will give a 22 attempt strategy for Greg. Let's form a table in which rows represent first digit and columns represent second digit. We cover the table by 22 figures, which represent Greg's moves. They consist of 9 crosses, 9 half-crosses, 2 dashes, and 2 dots. Note that one square (number 50) is not covered.

Now Greg's strategy is the following. He first tries numbers that are inside crosses. If all answers were "cold" then he tries numbers that are inside half-crosses. If again all answers were "cold" he tries numbers inside dashes. If the answers were "cold", he tries number that are dots. If all the answers were "cold", he knows that Alex's number was 50.

It remains to explain what to do if at some point Alex says "hot". Assume the answer is "hot" when Greg says a number inside a cross, say 23. Then in three more attempts Greg can find out Alex's number. Indeed, if 22 is "cold" and 24 is "hot", then Alex's number is 24. Similarly, if 22 is "hot" and 24 is "cold", then Alex's number is 22. If both are "cold" then if 13 is "cold", Alex's number is 33, otherwise it is 13. If both are "hot" the number is 23.

Assume now that the answer is "hot" when Greg says a number inside a half-cross or a dash. Then using similar arguments as before one can show that Greg can find out Alex's number in two more attempts. It might happen that the answer will be "hot" when Greg tries the number inside the last dash (that would be the 20th attempt). So in exactly 2 more attempts he will find out Alex's number!