March 4, 2001

TOURNAMENT OF TOWNS

Spring 2001, Level A, Junior (grades 8-10)

Your total score is based on the three problems for which you earn the most points; the scores for the individual parts of a single problem are summed. Points for each problem are shown in brackets [ ].

1. [3] In a certain country 10% of the employees get 90% of the total salary paid in this country. Supposing that the country is divided in several regions, is it possible that in every region the total salary of any 10% of the employees is no greater than 11% of the total salary paid in this region?

2. [5] In three piles there are 51, 49, and 5 stones, respectively. You can combine any two piles into one pile or divide a pile consisting of an even number of stones into two equal piles. Is it possible to get 105 piles with one stone in each?

3. [5] Point A lies inside an angle with vertex M. A ray issuing from point A is reflected in one side of the angle at point B, then in the other side at point C and then returns back to point A (the ordinary rule of reflection holds). Prove that the center of the circle circumscribed about triangle $\triangle BCM$ lies on line AM.

4. [5] Several non-intersecting diagonals divide a convex polygon into triangles. At each vertex of the polygon the number of triangles adjacent to it is written. Is it possible to reconstruct all the diagonals using these numbers if the diagonals are erased?

5.

(a) [3] One black and one white pawn are placed on a chessboard. You may move the pawns in turn to the neighbouring empty squares of the chessboard using vertical and horizontal moves. Can you arrange the moves so that every possible position of the two pawns will appear on the chessboard exactly once?

(b) [4] Same question, but you don't have to move the pawns in turn.

6. [7] Let AH_A, BH_B and CH_C be the altitudes of triangle $\triangle ABC$. Prove that the triangle whose vertices are the intersection points of the altitudes of triangles $\triangle AH_BH_C$, $\triangle BH_AH_C$ and $\triangle CH_AH_B$ is equal to triangle $\triangle H_AH_BH_C$.

7. Alex thinks of a two-digit integer (any integer between 10 and 99). Greg is trying to guess it. If the number Greg names is correct, or if one of its digits is equal to the corresponding digit of Alex's number and the other digit differs by one from the corresponding digit of Alex's number, then Alex says "hot"; otherwise, he says "cold". (For example, if Alex's number was 65, then by naming any of 64, 65, 66, 55 or 75 Greg will be answered "hot", otherwise he will be answered "cold".)

(a) [2] Prove that there is no strategy which guarantees that Greg will guess Alex's number in no more than 18 attempts.

(b) [3] Find a strategy for Greg to find out Alex's number (regardless of what the chosen number was) using no more than 24 attempts.

(c) [3] Is there a 22 attempt winning strategy for Greg?