A COURSE IN MATHEMATICAL ANALYSIS

FUNCTIONS OF A COMPLEX VARIABLE

BEING PART I OF VOLUME II

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FROM THE ORIGINAL AS PUBLISHED BY
GINN AND COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

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The Atheneum Press
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AUTHOR'S PREFACE – SECOND FRENCH EDITION

The first part of this volume has undergone only slight changes, while the rather important modifications that have been made appear only in the last chapters.

In the first edition I was able to devote but a few pages to partial differential equations of the second order and to the calculus of variations. In order to present in a less summary manner such broad subjects, I have concluded to defer them to a third volume, which will contain also a sketch of the recent theory of integral equations. The suppression of the last chapter has enabled me to make some additions, of which the most important relate to linear differential equations and to partial differential equations of the first order.

E. GOURSAT

TRANSLATOR'S PREFACE

As the title indicates, the present volume is a translation of the first half of the second volume of Goursat's "Cours d'Analyse." The decision to publish the translation in two parts is due to the evident adaptation of these two portions to the introductory courses in American colleges and universities in the theory of functions and in differential equations, respectively.

After the cordial reception given to the translation of Goursat's first volume, the continuation was assured. That it has been delayed so long was due, in the first instance, to our desire to await the appearance of the second edition of the second volume in French. The advantage in doing so will be obvious to those who have observed the radical changes made in the second (French) edition of the second volume. Volume I was not altered so radically, so that the present English translation of that volume may be used conveniently as a companion to this; but references are given here to both editions of the first volume, to avoid any possible difficulty in this connection.

Our thanks are due to Professor Goursat, who has kindly given us his permission to make this translation, and has approved of the plan of publication in two parts. He has also seen all proofs in English and has approved a few minor alterations made in translation as well as the translators' notes. The responsibility for the latter rests, however, with the translators.

E. R. HEDRICK
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CONTENTS

A COURSE IN
MATHEMATICAL ANALYSIS

VOLUME II. PART I

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

CHAPTER I

ELEMENTS OF THE THEORY

I. GENERAL PRINCIPLES. ANALYTIC FUNCTIONS

1. Definitions. An *imaginary quantity*, or *complex quantity*, is any expression of the form $a + bi$ where a and b are any two real numbers whatever and i is a special symbol which has been introduced in order to generalize algebra. Essentially a complex quantity is nothing but a system of two real numbers arranged in a certain order. Although such expressions as $a + bi$ have in themselves no concrete meaning whatever, we agree to apply to them the ordinary rules of algebra, with the additional convention that i^2 shall be replaced throughout by -1.

Two complex quantities $a + bi$ and $a' + b'i$ are said to be equal if $a = a'$ and $b = b'$. The sum of two complex quantities $a+bi$ and $c+di$ is a symbol of the same form $a+c+(b+d)i$; the difference $(a+bi)-(c+di)$ is equal to $a-c+(b-d)i$. To find the product of $a+bi$ and $c+di$ we carry out the multiplication according to the usual rules for algebraic multiplication, replacing i^2 by -1, obtaining thus

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

The quotient obtained by the division of $a+bi$ by $c+di$ is defined to be a third imaginary symbol $x+yi$, such that when it is multiplied by $c + di$, the product is $a + bi$. The equality

$$a + bi = (c + di)(x + yi)$$

is equivalent, according to the rules of multiplication, to the two relations

$$cx - dy = a, \qquad dx + cy = b,$$

whence we obtain

$$x = \frac{ac + bd}{c^2 + d^2}, \qquad y = \frac{bc - ad}{c^2 + d^2}.$$

[[4]] The quotient obtained by the division of $a + bi$ by $c + di$ is represented by the usual notation for fractions in algebra, thus, [[4]]

$$x + yi = \frac{a + bi}{c + di}.$$

A convenient way of calculating x and y is to multiply numerator and denominator of the fraction by $c - di$ and to develop the indicated products.

All the properties of the fundamental operations of algebra can be shown to apply to the operations carried out on these imaginary symbols. Thus, if A, B, C, $\cdots$ denote complex numbers, we shall have

$$A \cdot B = B \cdot A, \qquad A \cdot B \cdot C = A \cdot (B \cdot C), \qquad A(B + C) = AB + AC, \cdots$$

and so on. The two complex quantities $a + bi$ and $a - bi$ are said to be *conjugate imaginaries*. The two complex quantities $a+bi$ and $-a-bi$, whose sum is zero, are said to be *negatives* of each other or *symmetric* to each other.

Given the usual system of rectangular axes in a plane, the complex quantity $a+bi$ is represented by the point M of the plane xOy, whose coördinates are $x = a$ and $y = b$. In this way a concrete representation is given to these purely symbolic expressions, and to every proposition established for complex quantities there is a corresponding theorem of plane geometry. But the greatest advantages resulting from this representation will appear later. Real numbers correspond to points on the x-axis, which for this reason is also called the *axis of reals*. Two conjugate imaginaries $a + bi$ and $a - bi$ correspond to two points symmetrically situated with respect to the x-axis. Two quantities $a + bi$ and $-a - bi$ are represented by a pair of points symmetric with respect to the origin O. The quantity $a + bi$, which corresponds to the point M with the coördinates (a, b), is sometimes called its *affix*.* When there is no danger of ambiguity, we shall denote by the same letter a complex quantity and the point which represents it.

Let us join the origin to the point M with coördinates (a, b) by a segment of a straight line. The distance OM is called the *absolute value* of $a+bi$, and the angle through which a ray must be turned from Ox to bring it in coincidence with OM (the angle being measured, as in trigonometry, from Ox toward Oy) is called the
[[5]] *angle* of $a+bi$. Let ρ and ω denote, respectively, the absolute value and the angle of $a+bi$; between the real [[5]]
quantities a, b, ρ, ω there exist the two relations $a = \rho\cos\omega$, $b = \rho\sin\omega$, whence we have

$$\rho = \sqrt{a^2+b^2}, \qquad \cos\omega = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\omega = \frac{b}{\sqrt{a^2+b^2}}.$$

The absolute value ρ, which is an essentially positive number, is determined without ambiguity; whereas the angle, being given only by means of its trigonometric functions, is determined except for an additive multiple of 2π, which was evident from the definition itself. Hence every complex quantity may have an infinite number of angles, forming an arithmetic progression in which the successive terms differ by 2π. In order that two complex quantities be equal, their absolute values must be equal, and moreover their angles must differ only by a multiple of 2π, and these conditions are sufficient. The absolute value of a complex quantity z is represented by the same symbol $|z|$ which is used for the absolute value of a real quantity.

Let $z = a + bi$, $z' = a' + b'i$ be two complex numbers and m, m' the corresponding points; the sum $z + z'$ is then represented by the point m'', the vertex of the parallelogram constructed upon Om, Om'. The three sides of the triangle Omm'' (Fig. 1) are equal respectively to the absolute values of the quantities

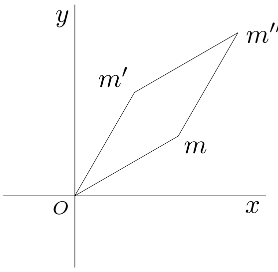

FIG. 1

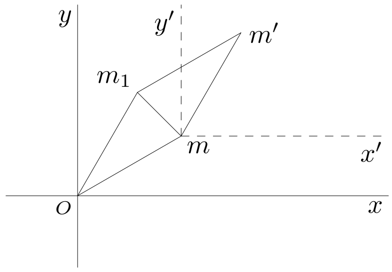

FIG. 2

z, z', $z + z'$. From this we conclude that *the absolute value of the sum of two quantities is less than or at most equal to the sum of the absolute values of the two quantities, and greater than or at least equal to their difference.* Since two quantities that are negatives of each other have the same absolute value, the theorem is also true for the absolute value of a difference. Finally, we see in the same way that the absolute value of the sum of any number of complex quantities is at most equal to the sum of their absolute values, the equality holding only when all the points representing the different equantities are on the same ray starting from the origin.

If through the point m we draw the two straight lines mx' and my' parallel to Ox and to Oy, the coördinates of the point m' in this system of axes will be $a' - a$ and $b' - b$ (Fig. 2). The point m' then
[[6]] represents $z' - z$ in the new system; the absolute value of $z' - z$ is equal to the length mm', and the angle of [[6]]
$z' - z$ is equal to the angle θ which the direction mm' makes with mx'. Draw through O a segment Om_1, equal and parallel to mm'; the extremity m_1 of this segment represents $z' - z$ in the system of axes Ox, Oy. But the figure $Om'm_1$ is a parallelogram; the point m_1 is therefore the symmetric point to m with respect to c, the middle point of Om'.

Finally, let us obtain the formula which gives the absolute value and angle of the product of any number of factors. Let

$$z_k = \rho_k(\cos\omega_k + i\sin\omega_k), \qquad (k = 1, 2, \cdots, n),$$

* This term is not much used in English, but the French frequently use the corresponding word *affixe*. – TRANS.

be the factors; the rules for multiplication, together with the addition formulæ of trigonometry, give for the product

$$z_1 z_2 \cdots z_n = \rho_1 \rho_2 \cdots \rho_n [\cos(\omega_1 + \omega_2 + \cdots + \omega_n) + i \sin(\omega_1 + \omega_2 + \cdots + \omega_n)],$$

which shows that *the absolute value of a product is equal to the product of the absolute values, and the angle of a product is equal to the sum of the angles of the factors.* From this follows very easily the well-known formula of De Moivre:

$$\cos m\omega + i \sin m\omega = (\cos\omega + i \sin\omega)^m,$$

which contains in a very condensed form all the trigonometric formulæ for the multiplication of angles.

The introduction of imaginary symbols has given complete generality and symmetry to the theory of algebraic equations. it was in the treatment of equations of only the second degree that such expressions appeared for the first time. Complex quantities are equally important in analysis, and we shall now state precisely what meaning is to be attached to the expression *a function of a complex variable.*

2. Continuous functions of a complex variable. A complex quantity $z = x + yi$, where x and y are two real and independent variables, is a complex variable. If we give to the word *function* its most general meaning, it would be natural to say that every other complex quantity u whose value depends upon
[[7]] that of z is a *function of z*. Certain familiar definitions can be extended directly to these functions. Thus, [[7]]
we shall say that a function $u = f(z)$ is continuous if the absolute value of the difference $f(z+h) - f(z)$ approaches zero when the absolute value of h approaches zero, that is, if to every positive number ϵ we can assign another positive number η such that

$$|f(z+h) - f(z)| < \epsilon,$$

provided that $|h|$ be less than η.

A series,

$$u_0(z) + u_1(z) + \cdots + u_n(z) + \cdots,$$

whose terms are functions of the complex variable z is *uniformly convergent* in a region A of the plane if to every positive number ϵ we can assign a positive integer N such that

$$|R_n| = |u_{n+1}(z) + u_{n+2}(z) + \cdots| < \epsilon$$

for all the values of z in the region A, provided that $n \geq N$. It can be shown as before (Vol. I, §31, 2d ed.; §173, 1st ed.) that if a series is uniformly convergent in a region A, and if each of its terms is a continuous function of z in that region, its sum is itself a continuous function of the variable z in the same region.

Again, a series is uniformly convergent if, for all the values of z considered, the absolute value of each term $|u_n|$ is less than the corresponding term v_n of a convergent series of real positive constants. The series is then both absolutely and uniformly convergent.

Every continuous function of the complex variable z is of the form $u = P(x,y) + Q(x,y)i$, where P and Q are real continuous functions of the two real variables x, y. If we were to impose no other restrictions, the study of functions of a complex variable would amount simply to a study of a pair of functions of two real variables, and the use of the symbol i would introduce only illusory simplifications. In order to make the theory of functions of a complex variable present some analogy with the theory of functions of a real variable, we shall adopt the methods of Cauchy to find the conditions which the functions P and Q must satisfy in order that the expression $P + Qi$ shall possess the fundamental properties of functions of a real variable to which the processes of the calculus apply.

3. Analytic functions. If $f(x)$ is a function of a real variable x which has a derivative, the quotient

$$\frac{f(x+h) - f(x)}{h}$$

[[8]] approaches $f'(x)$ when h approaches zero. Let us determine in the same way under what conditions the [[8]]
quotient

$$\frac{\Delta u}{\Delta z} = \frac{\Delta P + i\Delta Q}{\Delta x + i\Delta y}$$

will approach a definite limit when the absolute value of Δz approaches zero, that is, when Δx and Δy approach zero independently. It is easy to see that this will not be the case if the functions $P(x, y)$ and $Q(x, y)$ are any functions whatever, for the limit of the quotient $\Delta u/\Delta z$ depends in general on the ratio $\Delta y/\Delta x$, that is, on the way in which the point representing the value of $z+h$ approaches the point representing the value of z.

Let us first suppose y constant, and let us give to x a value $x + \Delta x$ differing but slightly from x; then

$$\frac{\Delta u}{\Delta z} = \frac{P(x + \Delta x, y) - P(x, y)}{\Delta x} + i\frac{Q(x + \Delta x, y) - Q(x, y)}{\Delta x}.$$

In order that this quotient have a limit, it is necessary that the functions P and Q possess partial derivatives with respect to x, and in that case

$$\lim \frac{\Delta u}{\Delta z} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}.$$

Next suppose x constant, and let us give to y the value $y + \Delta y$; we have

$$\frac{\Delta u}{\Delta z} = \frac{P(x, y + \Delta y) - P(x, y)}{i\Delta y} + \frac{Q(x, y + \Delta y) - Q(x, y)}{\Delta y},$$

and in this case the quotient will have for its limit

$$\frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}$$

if the functions P and Q possess partial derivatives with respect to y. In order that the limit of the quotient be the same in the two cases, it is necessary that

$$(1) \qquad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

Suppose that the functions P and Q satisfy these conditions, and that the partial derivatives $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, $\partial Q/\partial y$ are continuous functions. If we give to x and y any increments whatever, Δx, Δy, we can write

$$\begin{aligned}\Delta P &= P(x + \Delta x, y + \Delta y) - P(x + \Delta x, y) + P(x + \Delta x, y) - P(x, y)\\ &= \Delta y P'_y(x + \Delta x, y + \theta\Delta y) + \Delta x P'_x(x + \theta'\Delta x, y)\\ &= \Delta x[P'_x(x, y) + \epsilon] + \Delta y[P'_y(x, y) + \epsilon_1],\end{aligned}$$

[[9]] where θ and θ' are positive numbers less than unity; and in the same way [[9]]

$$\Delta Q = \Delta x[Q'_x(x, y) + \epsilon'] + \Delta y[Q'_y(x, y) + \epsilon'_1],$$

where ϵ, ϵ', ϵ_1, ϵ'_1 approach zero with Δx and Δy. The difference $\Delta u = \Delta P + i\Delta Q$ can be written by means of the conditions (1) in the form,

$$\begin{aligned}\Delta u &= \Delta x\left(\frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}\right) + \Delta y\left(-\frac{\partial Q}{\partial x} + i\frac{\partial P}{\partial x}\right) + \eta\Delta x + \eta'\Delta y\\ &= (\Delta x + i\Delta y)\left(\frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}\right) + \eta\Delta x + \eta'\Delta y,\end{aligned}$$

where η and η' are infinitesimals. We have, then,

$$\frac{\Delta u}{\Delta z} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} + \frac{\eta\Delta x + \eta'\Delta y}{\Delta x + i\Delta y}.$$

If $|\eta|$ and $|\eta'|$ are smaller than a number α, the absolute value of the complementary terms is less than 2α. This term will therefore approach zero when Δx and Δy approach zero, and we shall have

$$\lim \frac{\Delta u}{\Delta z} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}.$$

The conditions (1) are then necessary and sufficient in order that the quotient $\Delta u/\Delta z$ have a unique limit for each value of z, provided that the partial derivatives of the functions P and Q be continuous. The function u is then said to be an *analytic* function* of the variable z, and if we represent it by $f(z)$, the derivative $f'(z)$ is equal to any one of the following equivalent expressions:

$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} - i\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial y} + i\frac{\partial Q}{\partial x}. \tag{2}$$

It is important to notice that neither of the pair of functions $P(x, y)$, $Q(x, y)$ can be taken arbitrarily. In fact, if P and Q have derivatives of the second order, and if we differentiate the first of the relations (1) with respect to x, and the second with respect to y, we have, adding the two resulting equations,

$$\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

[[10]] We can show in the same way that $\Delta Q = 0$. The two functions $P(x, y)$, $Q(x, y)$ must therefore be a pair of [[10]] solutions of Laplace's equation.

Conversely, any solution of Laplace's equation may be taken for one of the functions P or Q. For example, let $P(x, y)$ be a solution of that equation; the two equations (1), where Q is regarded as an unknown function, are compatible, and the expression

$$u = P(x, y) + i\left[\int_{(x_0,y_0)}^{(x,y)} \left(\frac{\partial P}{\partial x}\,dy - \frac{\partial P}{\partial y}\,dx\right) + C\right],$$

which is determined except for an arbitrary constant C, is an analytic function whose real part is $P(x, y)$.

It follows that the study of analytic functions of a complex variable z amounts essentially to the study of a pair of functions $P(x, y)$, $Q(x, y)$ of two real variables x and y that satisfy the relations (1). It would be possible to develop the whole theory without making use of the symbol i.*

We shall continue, however, to employ the notation of Cauchy, but it should be noticed that there is no essential difference between the two methods. Every theorem established for an analytic function $f(z)$ can be expressed immediately as an equivalent theorem relating to the pair of functions P and Q, and conversely.

Examples. The function $u = x^2 - y^2 + 2xyi$ is an analytic function, for it satisfies the equations (1), and its derivative is $2x + 2yi = 2z$; in fact, the function is simply $(x + yi)^2 = z^2$. On the other hand, the expression $v = x - yi$ is not an analytic function, for we have

$$\frac{\Delta v}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \frac{1 - i\frac{\Delta y}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}},$$

and it is obvious that the limit of the quotient $\Delta v/\Delta z$ depends upon the limit of the quotient $\Delta y/\Delta x$.

If we put $x = \rho\cos\omega$, $y = \rho\sin\omega$, and apply the formulæ for the change of independent variables (I, §63, 2d ed.; §38, 1st ed., Ex. II), the relations (I) become

$$\frac{\partial P}{\partial \omega} = -\rho\frac{\partial Q}{\partial \rho}, \qquad \frac{\partial Q}{\partial \omega} = \rho\frac{\partial P}{\partial \rho}, \tag{3}$$

and the derivative takes the form

$$f'(z) = \left(\frac{\partial P}{\partial \rho} + i\frac{\partial Q}{\partial \rho}\right)(\cos\omega - i\sin\omega).$$

[[11]] It is easily seen on applying these formulæ that the function [[11]]

$$z^m = \rho^m(\cos m\omega + i\sin m\omega)$$

---

* Cauchy made frequent use of the term *monogène*, the equivalent of which, *monogenic*, is sometimes used in English. The term *synectique* is also sometimes used in French. We shall use by preference the term *analytic*, and it will be shown later that this definition agrees with the one which has already been given (I, §197, 2d ed.; §191, 1st ed.) [[The English term monogenic is no longer in common use.]]

* This is the point of view taken by the German mathematicians who follow Riemann.

is an analytic function of z whose derivative is equal to

$$m\rho^{m-1}(\cos m\omega + i \sin m\omega)(\cos\omega - i\sin\omega) = mz^{m-1}.$$

4. Functions analytic throughout a region. The preceding general statements are still somewhat vague, for so far nothing has been said about the limits between which z may vary.

A portion A of the plane is said to be *connected*, or to *consist of a single piece*, when it is possible to join any two points whatever of that portion by a continuous path which lies entirely in that portion of the plane. A connected portion situated entirely at a finite distance can be bounded by one or several closed curves, among which there is always one closed curve which forms the exterior boundary. A portion of the plane extending to infinity may be composed of all the points exterior to one or more closed curves; it may also be limited by curves having infinite branches. We shall employ the term *region* to denote a connected portion of the plane.

A function $f(z)$ of the complex variable z is said to be analytic* in a connected region A of the plane if it satisfies the following conditions:

1) To every point z of A corresponds a definite value of $f(z)$;

2) $f(z)$ is a continuous function of z when the point z varies in A, that is, when the absolute value of $f(z+h) - f(z)$ approaches zero with the absolute value of h;

3) At every point z of A, $f(z)$ has a uniquely determined derivative $f'(z)$; that is, to every point z corresponds a complex number $f'(z)$ such that the absolute value of the difference

$$\frac{f(z+h) - f(z)}{h} - f'(z)$$

approaches zero when $|h|$ approaches zero. Given any positive number ϵ, another positive number η can be found such that

$$|f(z+h) - f(z) - hf'(z)| \leq \epsilon|h| \tag{4}$$

if $|h|$ is less than η.

For the moment we shall not make any hypothesis as to the values of $f(z)$ on the curves which limit A.
[[12]] When we say that a function is analytic in the interior of a region A bounded by a closed curve Γ and *on the boundary curve itself*, we shall mean by this that $f(z)$ is analytic in a region $\mathcal{A}$ containing the boundary curve Γ and the region A. [[12]]

A function $f(z)$ need not necessarily be analytic throughout its region of existence. It may have, in general, singular points, which may be of very varied types. It would be out of place at this point to make a classification of these singular points, the very nature of which will appear as we proceed with the study of functions which we are now commencing.

5. Rational functions. Since the rules which give the derivative of a sum, of a product, and of a quotient are logical consequences of the definition of a derivative, they apply also to functions of a complex variable. The same is true of the rule for the derivative of a function of a function. Let $u = f(Z)$ be an analytic function of the complex variable Z; if we substitute for Z another analytic function $\phi(z)$ of another complex variable z, u is still an analytic function of the variable z. We have, in fact,

$$\frac{\Delta u}{\Delta z} = \frac{\Delta u}{\Delta Z} \times \frac{\Delta Z}{\Delta z};$$

when $|\Delta z|$ approaches zero, $|\Delta Z|$ approaches zero, and each of the quotients $\Delta u/\Delta Z$, $\Delta Z/\Delta z$ approaches a definite limit. Therefore the quotient $\Delta u/\Delta z$ itself approaches a limit:

$$\lim \frac{\Delta u}{\Delta z} = f'(Z)\phi'(z).$$

* The adjective *holomorphic* is also often used. – Trans.

We have already seen (§3) that the function

$$z^m = (x + yi)^m$$

is an analytic function of z, and that its derivative is mz^{m-1}. This can be shown directly as in the case of real variables. In fact, the binomial formula, which results simply from the properties of multiplication, obviously can be extended in the same way to complex quantities. Therefore we can write

$$(z+h)^m = z^m + \frac{m}{1} z^{m-1} h + \frac{m(m-1)}{1.2} z^{m-2} h^2 + \cdots,$$

where m is a positive integer; and from this follows

$$\frac{(z+h)^m - z^m}{h} = mz^{m-1} + h\left[\frac{m(m-1)}{1.2} z^{m-2} + \cdots + h^{m-2}\right].$$

[[13]] It is clear that the right-hand side has mz^{m-1} for its limit when the absolute value of h approaches zero. [[13]]

It follows that any polynomial with constant coefficients is an analytic function throughout the whole plane. A rational function (that is, the quotient of two polynomials $P(z)$, $Q(z)$, which we may as well suppose prime to each other) is also in general an analytic function, but it has a certain number of singular points, the roots of the equation $Q(z) = 0$. It is analytic in every region of the plane which does not include any of these points.

6. Certain irrational functions. When a point z describes a continuous curve, the coördinates x and y, as well as the absolute value ρ, vary in a continuous manner, and the same is also true of the angle, provided the curve described does not pass through the origin. If the point z describes a closed curve, x, y, and ρ return to their original values, but for the angle ω this is not always the case. If the origin is outside the region inclosed by the closed curve (Fig. 3*a*), it is evident that the angle will return to its original value; but this is no longer the case if the point z describes a curve such as M_0NPM_0 or M_0npqM_0 (Fig. 3*b*). In

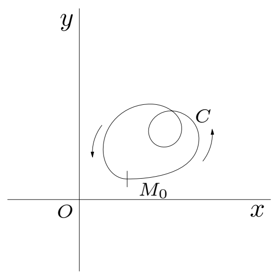

FIG. 3 *a*

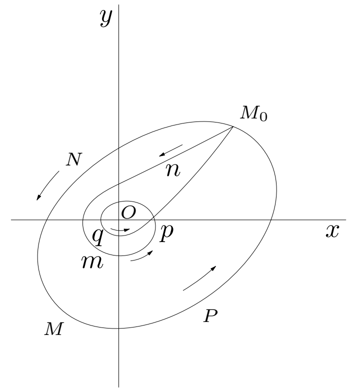

FIG. 3 *b*

the first case the angle takes on its original value increased by 2π, and in the second case it takes on its original value increased by 4π. It is clear that z can be made to describe closed curves such that, if we follow the continuous variation of the angle along any one of them, the final value assumed by ω will differ from the initial value by $2n\pi$, where n is an arbitrary integer, positive or negative. In general, when z describes
[[14]] a closed curve, the angle of $z - a$ returns to its initial value if the point a lies outside of the region bounded [[14]]
by that closed curve, but the curve described by z can always be chosen so that the final value assumed by the angle of $z - a$ will be equal to the initial value increased by $2n\pi$.

Let us now consider the equation

$$u^m = z, \tag{5}$$

where m is a positive integer. To every value of z, except $z = 0$, there are m distinct values of u which satisfy this equation and therefore correspond to the given value of z. In fact, if we put

$$z = \rho(\cos\omega + i\sin\omega), \qquad u = r(\cos\phi + i\sin\phi),$$

the relation (5) becomes equivalent to the following pair:

$$r^m = \rho, \qquad m\phi = \omega + 2k\pi.$$

From the first we have $r = \rho^{1/m}$, which means that r is the mth arithmetic root of the positive number ρ; from the second we have

$$\phi = (\omega + 2k\pi)/m.$$

To obtain all the distinct values of u we have only to give to the arbitrary integer k the m consecutive integral values $0, 1, 2, \cdots, m-1$; in this way we obtain expressions for the m roots of the equation (5)

$$u_k = \rho^{\frac{1}{m}} \left[\cos\left(\frac{\omega + 2k\pi}{m}\right) + i \sin\left(\frac{\omega + 2k\pi}{m}\right)\right], \qquad (k = 0, 1, 2, \cdots, m-1). \tag{6}$$

It is usual to represent by $z^{1/m}$ any one of these roots.

When the variable z describes a continuous curve, each of these roots itself varies in a continuous manner. If z describes a closed curve to which the origin is exterior, the angle ω comes back to its original value, and each of the roots $u_0, u_1, \cdot, u_{m-1}$ describes a closed curve (Fig. 4*a*). But if the point z describes the curve M_0NPM_0 (Fig. 3*b*), ω changes to $\omega + 2\pi$, and the final value of the root u_i is equal to the initial value of the root u_{i+1}. Hence the arcs described by the different roots form a single closed curve (Fig. 4*b*).

These m roots therefore undergo a cyclic permutation when the variable z describes in the positive direction any closed curve without double points that incloses the origin. It is clear that by making z describe a suitable closed path, any one of the roots, starting from the initial value u_0, for example, can be made to take on for its final value the value of any of the other roots. If we wish to maintain continuity, [[15]] we must then consider these m roots of the equation (5) not as so many distinct functions of z, but as m [[15]] distinct *branches* of the same function. The point $z = 0$, about which the permutation of the m values of u takes place, is called a *critical point* or a *branch point*.

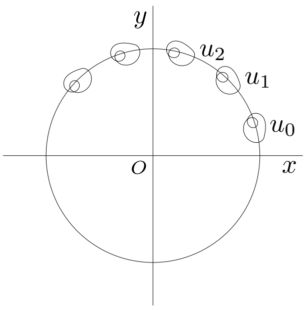

FIG. 4 *a*

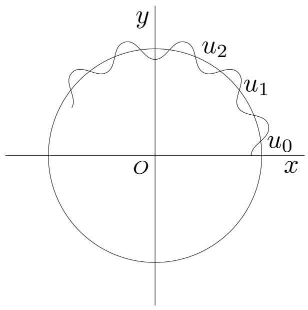

FIG. 4 *b*

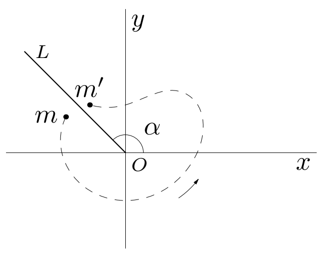

FIG. 5

In order to consider the m values of u as distinct functions of z, it will be necessary to disrupt the continuity of these roots along a line proceeding from the origin to infinity. We can represent this break in the continuity very concretely as follows: imagine that in the plane of z, which we may regard as a thin sheet, a cut is made along a ray extending from the origin to infinity, for example, along the ray OL (Fig. 5), and that then the two edges of the cut are slightly separated so that there is no path along which the variable z can move directly from one edge to the other. Under these circumstances no closed path whatever can inclose the origin; hence to each value of z corresponds a completely determined value u_i of the m roots, which we can obtain by taking for the angle ω the value included between α and $\alpha - 2\pi$. But it must be noticed that the values of u_i at two points m, m' on opposite sides of the cut do not approach the same limit as the points approach the same point of the cut. The limit of the value of u_i at the point m' is equal to the limit of the value of u_i at the point m, multiplied by $[\cos(2\pi/m) + i\sin(2\pi/m)]$.

Each of the roots of the equation (5) is an analytic function. Let u_0 be one of the roots corresponding [[16]] to a given value z_0; to a value of z near z_0 corresponds a value of u near u_0. Instead of trying to find the [[16]] limit of the quotient $(u - u_0)/(z - z_0)$, we can determine the limit of its reciprocal

$$\frac{z - z_0}{u - u_0} = \frac{u^m - u_0^m}{u - u_0},$$

and that limit is equal to mu_0^{m-1}. We have, then, for the derivative of u

$$u' = \frac{1}{m}\frac{1}{u^{m-1}} = \frac{1}{m}\frac{u}{z},$$

or, using negative exponents,

$$u' = \frac{1}{m} z^{\frac{1}{m}-1}.$$

In order to be sure of having the value of the derivative which corresponds to the root considered, it is better to make use of the expression $(1/m)(u/z)$.

In the interior of a closed curve not containing the origin each of the determinations of $\sqrt[m]{z}$ is an analytic function. The equation $u^m = A(z-a)$ has also m roots, which permute themselves cyclically about the critical point $z = a$.

Let us consider now the equation

$$u^2 = A(z-e_1)(z-e_2)\cdots(z-e_n), \tag{7}$$

where $e_1, e_2, \cdots, e_n$ are n distinct quantities. We shall denote by the same letters the points which represent these n quantities. Let us set

$$\begin{aligned} A &= R(\cos\alpha + i\sin\alpha), \\ z - e_k &= \rho_k(\cos\omega_k + i\sin\omega_k), \qquad (k = 1, 2, \cdots, n), \\ u &= r(\cos\theta + i\sin\theta), \end{aligned}$$

where ω_k represents the angle which the straight-line segment $e_k z$ makes with the direction Ox. From the equation (7) it follows that

$$r^2 = R\rho_1\rho_2\cdots\rho_n, \qquad 2\theta = \alpha + \omega_1 + \cdots + \omega_n + 2m\pi;$$

hence this equation has two roots that are the negatives of each other,

$$\left\{\begin{aligned} u_1 &= (R\rho_1\rho_2\cdots\rho_n)^{\frac{1}{2}}\left[\cos\left(\frac{\alpha+\omega_1+\cdots+\omega_n}{2}\right) + i\sin\left(\frac{\alpha+\omega_1+\cdots+\omega_n}{2}\right)\right], \\ u_2 &= (R\rho_1\rho_2\cdots\rho_n)^{\frac{1}{2}}\left[\cos\left(\frac{\alpha+\omega_1+\cdots+\omega_n+2\pi}{2}\right) + i\sin\left(\frac{\alpha+\omega_1+\cdots+\omega_n+2\pi}{2}\right)\right]. \end{aligned}\right. \tag{8}$$

[[17]] When the variable z describes a closed curve C containing within it p of the points $e_1, e_2, \cdots, e_n$, p of the angles $\omega_1, \omega_2, \cdots, \omega_n$ will increase by 2π; the angle of u_1 and that of u_2 will therefore increase by $p\pi$. If p is even, the two roots return to their initial values; but if p is odd, they are permuted. In particular, if the curve incloses a single point e_i, the two roots will be permuted. The n points e_i are branch points. In order that the two roots u_1 and u_2 shall be functions of z that are always uniquely determined, it will suffice to make a system of cuts such that any closed curve whatever will always contain an even number of critical points. We might, for example, make cuts along rays proceeding from each of the points e_i to infinity and not cutting each other. But there are many other possible arrangements. If, for example, there are four critical points e_1, e_2, e_3, e_4, a cut could be made along the segment of a straight line e_1e_2, and a second along the segment e_3e_4. [[17]]

7. Single-valued and multiple-valued functions. The simple examples which we have just treated bring to light a very important fact. The value of a function $f(z)$ of the variable z does not always depend entirely upon the value of z alone, but it may also depend in a certain measure upon the succession of values assumed by the variable z in passing from the initial value to the actual value in question, or, in other words, upon the path followed by the variable z.

Let us return, for example, to the function $u = \sqrt[m]{z}$. If we pass from the point M_0 to the point M by the two paths M_0NM and M_0PM (Fig. 3*b*), starting in each case with the same initial value for u, we shall not obtain at M the same value for u, for the two values obtained for the angle of z will differ by 2π. We are thus led to introduce a new distinction.

An analytic function $f(z)$ is said to be *single-valued** in a region A when all the paths in A which go from a point z_0 to any other point whatever z lead to the same final value for $f(z)$. When, however, the final value of $f(z)$ is not the same for all possible paths in A, the function is said to be *multiple-valued.*† A function that is analytic at every point of a region A is necessarily single-valued in that region. In general, in order that a function $f(z)$ be single-valued in a given region, it is necessary and sufficient that the function [[18]] return to its original value when the variable makes a circuit of any closed path whatever. If, in fact, in going from the point A to the point B by the two paths AMB (Fig. 6) and ANB, we arrive in the two cases at the point B with the same determination of $f(z)$, it is obvious that, when the variable is made to describe the closed curve $AMBNA$, we shall return to the point A with the initial value of $f(z)$.

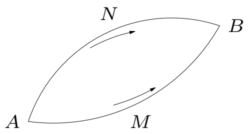

Fig. 6

Conversely, let us suppose that, the variable having described the path $AMBNA$, we return to the point of departure with the initial value u_0; and let u_1 be the value of the function at the point B after z has described the path AMB. When z describes the path BNA, the function starts with the value u_1 and arrives at the value u_0; then, conversely, the path ANB will lead from the value u_0 to the value u_1, that is, to the same value as the path AMB.

It should be noticed that a function which is not single-valued in a region may yet have no critical points in that region. Consider, for example, the portion of the plane included between two concentric circles C, C' having the origin for center. The function $u = z^{1/m}$ has no critical point in that region; still it is not single-valued in that region, for if z is made to describe a concentric circle between C and C', the function $z^{1/m}$ will be multiplied by $\cos(2\pi/m) + i\sin(2\pi/m)$.

II. POWER SERIES WITH COMPLEX TERMS. ELEMENTARY TRANSCENDENTAL FUNCTIONS

8. Circle of convergence. The reasoning employed in the study of power series (Vol. I, Chap. IX) will apply to power series with complex terms; we have only to replace in the reasoning the phrase "absolute value of a real quantity" by the corresponding one, "absolute value of a complex quantity." We shall recall briefly the theorems and results stated there. Let

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{9}$$

be a power series in which the coefficients and the variable may have any imaginary values whatever. Let us also consider the series of absolute values,

$$A_0 + A_1 r + A_2 r^2 + \cdots + A_n r^n + \cdots, \tag{10}$$

where $A_i = |a_i|$, $r = |z|$. We can prove (I, §181, 2d ed.; §177, 1st ed.) the existence of a positive number R [[19]] such that the series (10) is convergent for every value of $r < R$, and divergent for every value of $r > R$. The number R is equal to the reciprocal of the greatest limit of the terms of the sequence

$$A_1, \quad \sqrt[2]{A_2}, \quad \sqrt[3]{A_3}, \quad \cdots, \quad \sqrt[n]{A_n}, \quad \cdots,$$

and, as particular cases, it may be zero or infinite.

From these properties of the number R it follows at once that the series (9) is absolutely convergent when the absolute value of z is less than R. It cannot be convergent for a value z_0 of z whose absolute value is greater than R, for the series of absolute values (10) would then be convergent for values of r greater than R (I, §181, 2d ed.; §177, 1st ed.). If, with the origin as center, we describe in the plane of the variable z a

* In French the term *uniforme* or the term *monodrome* is used. – Trans.

† In French the term *multiforme* is used. – Trans.

circle C of radius R (Fig. 7), the power series (9) is absolutely convergent for every value of z inside the circle

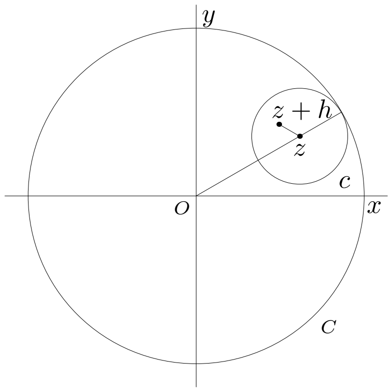

FIG. 7

C, and divergent for every value of z outside; for this reason the circle is called the *circle of convergence*. In a point of the circle itself the series may be convergent or divergent, according to the particular series.*

In the interior of a circle C' concentric with the first, and with a radius R' less than R, the series (9) is uniformly convergent. For at every point within C' we have evidently

$$|a_{n+1}z^{n+1} + \cdots + a_{n+p}z^{n+p}| < A_{n+1}R'^{n+1} + \cdots + A_{n+p}R'^{n+p},$$

and it is possible to choose the integer n so large that the second member will be less than any given positive number ϵ, whatever p may be. From this we conclude that the sum of the series (9) is a continuous function $f(z)$ of the variable z at every point within the circle of convergence (§2).

By differentiating the series (9) repeatedly, we obtain an unlimited number of power series, $f_1(z)$, $f_2(z)$, $\cdots$, $f_n(z)$, $\cdots$, which have the same circle of convergence as the first (I, §183, 2d ed.; §179, 1st ed.). We prove in the same way as in §184, 2d ed., that $f_1(z)$ is the derivative of $f(z)$, and in general that $f_n(z)$ is
[[20]] the derivative of $f_{n-1}(z)$. *Every power series represents therefore an analytic function in the interior of its* [[20]]
circle of convergence. There is an infinite sequence of derivatives of the given function, and all of these are analytic functions in the same circle. Given a point z inside the circle C, let us draw a circle c tangent to the circle C in the interior, with the given point as center, and then let us take a point $z + h$ inside c; if r and ρ are the absolute values of z and h, we have $r + \rho < R$ (Fig. 7). The sum $f(z + h)$ of the series is equal to the sum of the double series

$$(11) \qquad \left\{\begin{array}{l} a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \\ \quad + a_1 h + 2a_2 zh + \cdots + na_n z^{n-1}h + \cdots \\ \qquad\quad + a_2 h^2 + \cdots + \dfrac{n(n-1)}{1.2} a_n z^{n-2} h^2 + \cdots \\ \qquad\quad + \cdots \end{array}\right.$$

when we sum by columns. But this series is absolutely convergent, for if we replace each term by its absolute value, we shall have a double series of positive terms whose sum is

$$A_0 + A_1(r + \rho) + \cdots + A_n(r + \rho)^n + \cdots.$$

* Let $f(x) = \sum a_n z^n$ be a power series whose radius of convergence R is equal to 1. If the coefficients $a_0, a_1, a_2, \cdots$, are positive decreasing numbers such that a_n approaches zero when n increases indefinitely, the series is convergent in every point of the circle of convergence, except perhaps for $z = 1$. In fact, the series $\sum z^n$, where $|z| = 1$, is indeterminate except for $z = 1$, for the absolute value of the sum of the first n terms is less than $2/|1 - z|$; it will suffice, then to apply the reasoning of §166, Vol. I, based on the generalized lemma of Abel. In the same way the series $a_0 - a_1 z + a_2 z^2 - \cdots$, which is obtained from the preceding by replacing z by $-z$, is convergent at all the points of the circle $|z| = 1$, except perhaps for $z = -1$. (Cf. I, §166.)

We can therefore sum the double series (11) by rows, and we have then, for every point $z + h$ inside the circle c, the relation

$$f(z+h) = f(z) + hf_1(z) + \frac{h^2}{2!}f_2(z) + \cdots + \frac{h^n}{n!}f_n(z) + \cdots. \tag{12}$$

The series of the second member is surely convergent so long as the absolute value of h is less than $R - r$, but it may be convergent in a larger circle. Since the functions $f_1(z)$, $f_2(z)$, $\cdots$, $f_n(z)$, $\cdots$ are equal to the successive derivatives of $f(z)$, the formula (12) is identical with the Taylor development.

If the series (9) is convergent at a point Z of the circle of convergence, the sum $f(Z)$ of the series is the
[[21]] limit approached by the sum $f(z)$ when the point z approaches the point Z along a radius which terminates [[21]]
in that point. We prove this just as in Volume I (§182, 2d ed.; §178, 1st ed.), by putting $z = \theta Z$ and letting θ increase from 0 to 1. The theorem is still true when z, remaining inside the circle, approaches Z along a curve which is not tangent at Z to the circle of convergence.*

When the radius R is infinite, the circle of convergence includes the whole plane, and the function $f(z)$ is analytic for every value of z. We say that this is an *integral function*; the study of transcendental functions of this kind is one of the most important objects of Analysis.† We shall study in the following paragraphs the classic elementary transcendental functions.

9. Double series. Given a power series (9) with any coefficients whatever, we shall say again that a second power series $\sum \alpha_n z^n$, whose coefficients are all real and positive, *dominates* the first series if for every value of n we have $|a_n| \leq \alpha_n$. All the consequences deduced by means of dominant functions (I, §§186 – 189, 2d ed.; §§181 – 184, 1st ed.) follow without modification in the case of complex variables. We shall now give another application of this theory.

Let

$$f_0(z) + f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots \tag{13}$$

be a series of which each term is itself the sum of a power series that converges in a circle of radius equal to or greater than the number $R > 0$,

$$f_i(z) = a_{i0} + a_{i1}z + \cdots + a_{in}z^n + \cdots.$$

Suppose each term of the series (13) replaced by its development according to powers of z; we obtain thus a double series in which each column is formed by the development of a function $f_i(z)$. When that series is absolutely convergent for a value of z of absolute value ρ, that is, when the double series

$$\sum_i \sum_n |a_{in}|\rho^n$$

is convergent, we can sum the first double series by rows for every value of z whose absolute value does not exceed ρ. We obtain thus the development of the sum $F(z)$ of the series (13) in powers of z,

$$\begin{aligned} F(z) &= b_0 + b_1 z + \cdots + b_n z^n + \cdots, \\ b_n &= a_{0n} + a_{1n} + \cdots + a_{in} + \cdots, \qquad (n = 0, 1, 2, \cdots). \end{aligned}$$

This proof is essentially the same as that for the development of $f(z+h)$ in powers of h.

Suppose, for example, that the series $f_i(z)$ has a dominant function of the form $M_i r/(r - z)$, and that the series $\sum M_i$
[[22]] is itself convergent. In the double series the absolute value of the general term is less than $M_i|z|^n/r^n$. If $|z| < r$, the series is [[22]]
absolutely convergent, for the series of the absolute values is convergent and its sum is less than $r \sum M_i/(r - |z|)$.

10. Development of an infinite product in power series. Let

$$F(z) = (1 + u_0)(1 + u_1)\cdots(1 + u_n)\cdots$$

be an infinite product where each of the functions u_i is a continuous function of the complex variable z in the region D. If the series $\sum U_i$, where $U_i = |u_i|$, is uniformly convergent in the region, $F(z)$ is equal to the sum of a series that is uniformly convergent in D, and therefore represents a continuous function (I, §§175, 176, 2d ed.). When the functions u_i are analytic functions of z, it follows, from a general theorem which will be demonstrated later (§39), that the same is true of $F(z)$.

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* See PICARD, *Traité d'Analyse*, Vol. II, p. 73.

† The class of *integral functions* includes polynomials as a special case. If there are an infinite number of terms in the development, we shall use the expression *integral transcendental function.* – TRANS.

For example, the infinite product

$$F(z) = z(1 - z^2)\left(1 - \frac{z^2}{4}\right)\cdots\left(1 - \frac{z^2}{n^2}\right)\cdots$$

represents a function of z analytic throughout the entire plane, for the series $\sum |z|^2/n^2$ is uniformly convergent within any closed curve whatever. This product is zero for $z = 0, \pm 1, \pm 2, \cdots$ and for these values only.

We can prove directly that the product $F(z)$ can be developed in a power series when each of the functions u_i can be developed in a power series

$$u_i(z) = a_{i0} + a_{i1}z + \cdots + a_{in}z^n + \cdots, \qquad (i = 0, 1, 2, \cdots),$$

such that the double series

$$\sum_i \sum_n |a_{in}|\, r^n$$

is convergent for a suitably chosen positive value of r.

Let us set, as in Volume I (§174, 2d ed.),

$$v_0 = 1 + u_0, \qquad v_n = (1 + u_0)(1 + u_1)\cdots(1 + u_{n-1})u_n.$$

It is sufficient to show that the sum of the series

$$v_0 + v_1 + \cdots + v_n + \cdots, \tag{14}$$

which is equal to the infinite product $F(z)$, can be developed in a power series. Now, if we set

$$u_i' = |a_{i0}| + |a_{i1}|\, z + \cdots + |a_{in}|\, z^n + \cdots,$$

it is clear that the product

$$v_n' = (1 + u_0')(1 + u_1')\cdots(1 + u_{n-1}')u_n'$$

is a dominant function for v_n. It is therefore possible to arrange the series (14) according to powers of z if the following auxiliary series

$$v_0' + v_1' + \cdots + v_n' + \cdots \tag{15}$$

can be so arranged.

If we develop each term of this last series in power series, we obtain a double series with positive coefficients, and it is
[[23]] sufficient for our purpose to prove that the double series converges when z is replaced by r. Indicating by U_n' and V_n' the values [[23]]
of the functions u_n' and v_n' for $z = r$, we have

$$V_n' = (1 + U_0')(1 + U_1')\cdots(1 + U_{n-1}')U_n',$$

and therefore

$$V_0' + V_1' + \cdots + V_n' = (1 + U_0')\cdots(1 + U_n'),$$

or, again,

$$V_0' + V_1' + \cdots + V_n' < e^{U_0' + \cdots + U_n'}.$$

When n increases indefinitely, the sum $U_0' + \cdots + U_n'$ approaches a limit, since the series $\sum U_n'$ is supposed to be convergent. The double series (15) is then absolutely convergent if $|z| \le r$; the double series obtained by the development of each term v_n of the series (14) is then a fortiori absolutely convergent within the circle C of radius r, and we can arrange it according to integral powers of z.

The coefficient b_p of z^p in the development of $F(z)$ is equal, from the above, to the limit, as n becomes infinite, of the coefficient b_{pn} of z^p in the sum $v_0 + v_1 + \cdots + v_n$, or, what amounts to the same thing, in the development of the product

$$P_n = (1 + u_0)(1 + u_1)\cdots(1 + u_n).$$

Hence this coefficient can be obtained by applying to infinite products the ordinary rule which gives the coefficient of a power of z in the product of a finite number of polynomials. For example, the infinite product

$$F(z) = (1 + z)(1 + z^2)(1 + z^4)\cdots(1 + z^{2^n})\cdots$$

can be developed according to powers of z if $|z| < 1$. Any power of z whatever, say z^N, will appear in the development with the coefficient unity, for any positive integer N can be written in one and only one way in the form of a sum of powers of 2. We have, then, if $|z| < 1$,

$$F(z) = 1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1-z}, \tag{16}$$

which can also be very easily obtained by means of the identity

$$\frac{1 - z^{2^n}}{1 - z} = (1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^{n-1}}).$$

11. The exponential function. The arithmetic definition of the exponential function evidently has no meaning when the exponent is a complex number. In order to generalize the definition, it will be necessary to start with some property which is adapted to an extension to the case of the complex variable. We shall start with the property expressed by the functional relation

$$a^x \times a^{x'} = a^{x+x'}.$$

Let us consider the question of determining a power series $f(z)$, convergent in a circle of radius R, such that

$$f(z + z') = f(z)f(z') \tag{17}$$

[[24]] when the absolute values of z, z', $z + z'$ are less than R, which will surely be the case if $|z|$ and $|z'|$ are less [[24]]
than $R/2$. If we put $z' = 0$ in the above equation, it becomes

$$f(z) = f(z)f(0).$$

Hence we must have $f(0) = 1$, and we shall write the desired series

$$f(z) = 1 + \frac{a_1}{1}z + \frac{a_2}{2!}z^2 + \cdots + \frac{a_n}{n!}z^n + \cdots.$$

Let us replace successively in that series z by λt, then by $\lambda' t$, where λ and λ' are two constants and t an auxiliary variable; and let us then multiply the resulting series. This gives

$$\begin{aligned} f(\lambda t)f(\lambda' t) = 1 &+ \frac{a_1}{1}(\lambda + \lambda')t + \cdots \\ &+ \frac{t^n}{n!}\left(a_n\lambda^n + \frac{n}{1}a_{n-1}a_1\lambda^{n-1}\lambda' + \cdots + a_n\lambda'^n\right) + \cdots. \end{aligned}$$

On the other hand, we have

$$f(\lambda t + \lambda' t) = 1 + \frac{a_1}{1}(\lambda + \lambda')t + \cdots + \frac{a_n}{n!}(\lambda + \lambda')^n t^n + \cdots.$$

The equality $f(\lambda t + \lambda' t) = f(\lambda t)f(\lambda' t)$ is to hold for all values of λ, λ', t such that $|\lambda| < 1$, $|\lambda'| < 1$, $|t| < R/2$. The two series must then be identical, that is, we must have

$$\begin{aligned} a_n(\lambda + \lambda')^n = a_n\lambda^n &+ \frac{n}{1}a_{n-1}a_1\lambda^{n-1}\lambda' \\ &+ \frac{n(n-1)}{1.2}a_{n-2}a_2\lambda^{n-2}\lambda'^2 + \cdots + a_n\lambda'^n, \end{aligned}$$

and from this we can deduce the equations

$$a_n = a_{n-1}a_1, \qquad a_n = a_{n-2}a_2, \qquad \cdots,$$

all of which can be expressed in the single condition

$$a_{p+q} = a_p a_q, \tag{18}$$

where p and q are any two positive integers whatever. In order to find the general solution, let us suppose $q = 1$, and let us put successively $p = 1$, $p = 2$, $p = 3$, $\cdots$; from this we find $a_2 = a_1^2$, then $a_3 = a_2 a_1 + a_1^3$, $\cdots$, and finally $a_n = a_1^n$. The expressions thus obtained satisfy the condition (18), and the series sought is of the form

$$f(z) = 1 + \frac{a_1 z}{1} + \frac{(a_1 z)^2}{2!} + \cdots + \frac{(a_1 z)^n}{n!} + \cdots.$$

[[25]] This series is convergent in the whole plane, and the relation [[25]]

$$f(z + z') = f(z) f(z')$$

is true for all values of z and z'.

The above series depends upon an arbitrary constant a_1. Taking $a_1 = 1$, we shall set

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots,$$

so that the general solution of the given problem is $e^{a_1 z}$. The integral function e^z coincides with the exponential function e^x studied in algebra when z is real, and it always satisfies the relation

$$e^{z+z'} = e^z \times e^{z'},$$

whatever z and z' may be. The derivative of e^z is equal to the function itself. Since we may write by the addition formula

$$e^{x+yi} = e^x e^{yi},$$

in order to calculate e^z when z has an imaginary value $x + yi$, it is sufficient to know how to calculate e^{yi}. Now the development of e^{yi} can be written, grouping together terms of the same kind,

$$e^{yi} = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots + i\left(\frac{y}{1} - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right).$$

We recognize in the second member the developments of $\cos y$ and of $\sin y$, and consequently, if y *is real*,

$$e^{yi} = \cos y + i \sin y.$$

Replacing e^{yi} by this expression in the preceding formula, we have

$$e^{x+yi} = e^x (\cos y + i \sin y); \tag{19}$$

the *function* e^{x+yi} *has* e^x *for its absolute value and* y *for its angle.* This formula makes evident an important property of e^z; if z changes to $z + 2\pi i$, x is not changed while y is increased by 2π, but these changes do not alter the value of the second member of the formula (19). We have, then,

$$e^{z+2\pi i} = e^z;$$

that is, *the exponential function* e^z *has the period* $2\pi i$.

Let us consider now the solution of the equation $e^z = A$, where A is any complex quantity whatever different fro zero. Let ρ and ω be the absolute value and the angle of A; we have, then,

$$e^{x+yi} = e^x (\cos y + i \sin y) = \rho(\cos \omega + i \sin \omega),$$

[[26]] from which it follows that [[26]]

$$e^x = \rho, \qquad y = \omega + 2k\pi.$$

From the first relation we find $x = \log\rho$, where the abbreviation *log* shall always be used for the natural logarithm of a real positive number. On the other hand, y is determined except for a multiple of 2π. If A is zero, the equation $e^x = 0$ leads to an impossibility. Hence *the equation* $e^z = A$, *where* A *is not zero, has an infinite number of roots given by the expression* $\log\rho + i(\omega + 2k\pi)$; *the equation* $e^z = 0$ *has no roots, real or imaginary.*

Note. We might also define e^z as the limit approached by the polynomial $(1 + z/m)^m$ when m becomes infinite. The method used in algebra to prove that the limit of this polynomial is the series e^z can be used even when z is complex.

12. Trigonometric functions. In order to define $\sin z$ and $\cos z$ when z is complex, we shall extend directly to complex values the series established for these functions when the variable is real. Thus we shall have

$$\left\{\begin{aligned} \sin z &= \frac{z}{1} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots. \end{aligned}\right. \tag{20}$$

These are integral transcendental functions which have all the properties of the trigonometric functions. Thus we see from the formulæ (20) that the derivative of $\sin z$ is $\cos z$, that the derivative of $\cos z$ is $-\sin z$, and that $\sin z$ becomes $-\sin z$, while $\cos z$ does not change at all when z is changed to $-z$.

These new transcendental functions can be brought into very close relation with the exponential function. In fact, if we write the expansion of e^{zi}, collecting separately the terms with and without the factor i,

$$e^{zi} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + i\left(\frac{z}{1} - \frac{z^3}{3!} + \cdots\right),$$

we find that the equality can be written, by (20), in the form

$$e^{zi} = \cos z + i\sin z.$$

Changing z to $-z$, we have again

$$e^{-zi} = \cos z - i\sin z,$$

and from these two relations we derive

$$\cos z = \frac{e^{zi} + e^{-zi}}{2}, \qquad \sin z = \frac{e^{zi} - e^{-zi}}{2i}. \tag{21}$$

[[27]] These are the well-known formulæ of Euler which express the trigonometric functions in terms of the exponential function. They show plainly the periodicity of these functions, for the right-hand sides do not change when we replace z by $z + 2\pi$. Squaring and adding them, we have [[27]]

$$\cos^2 z + \sin^2 z = 1.$$

Let us take again the addition formula $e^{(z+z')i} = e^{zi}e^{z'i}$, or

$$\begin{aligned} \cos(z + z') + i\sin(z + z') &= (\cos z + i\sin z)(\cos z' + i\sin z') \\ &= \cos z\cos z' - \sin z\sin z' + i(\sin z\cos z' + \sin z'\cos z), \end{aligned}$$

and let us change z to $-z$, z' to $-z'$. It then becomes

$$\cos(z + z') - i\sin(z + z') = \cos z\cos z' - \sin z\sin z' - i(\sin z\cos z' + \sin z'\cos z),$$

and from these two formulæ we derive

$$\begin{aligned} \cos(z + z') &= \cos z\cos z' - \sin z\sin z' \\ \sin(z + z') &= \sin z\cos z' + \sin z\cos z'. \end{aligned}$$

The addition formulæ and therefore all their consequences apply for complex values of the independent variables. Let us determine, for example, the real part and the coefficient of i in $\cos(x+yi)$ and $\sin(x+yi)$. We have first, by Euler's formulæ,

$$\cos yi = \frac{e^{-y}+e^{y}}{2} = \cosh y, \qquad \sin yi = \frac{e^{-y}-e^{y}}{2i} = i\sinh y;$$

whence, by the addition formulæ,

$$\cos(x+yi) = \cos x\cos yi - \sin x\sin yi = \cos x\cosh y - i\sin x\sinh y,$$
$$\sin(x+yi) = \sin x\cos yi + \cos x\sin yi = \sin x\cosh y + i\cos x\sinh y.$$

The other trigonometric functions can be expressed by means of the preceding. For example,

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{zi}-e^{-zi}}{e^{zi}+e^{-zi}},$$

which may be written in the form

$$\tan z = \frac{1}{i}\frac{e^{2zi}-1}{e^{2zi}+1}.$$

The right-hand side is a rational function of e^{2zi}; the period of the tangent is therefore π.

[[28]] **13. Logarithms.** Given a complex quantity z, different from zero, we have already seen (§11) that [[28]]
the equation $e^u = z$ has an infinite number of roots. Let $u = x + yi$, and let ρ and ω denote the absolute value and angle of z, respectively. Then we must have

$$e^x = \rho, \qquad y = \omega + 2k\pi.$$

Any one of these roots is called the *logarithm* of z and will be denoted by $\mathrm{Log}\,(z)$. We can write, then,

$$\mathrm{Log}\,(z) = \log\rho + i(\omega + 2k\pi),$$

the symbol *log* being reserved for the ordinary natural, or Napierian, logarithm of a real positive number.

Every quantity, real or complex, different from zero, has an infinite number of logarithms, which form an arithmetic progression whose consecutive terms differ by $2\pi i$. In particular, if z is a real positive number x, we have $\omega = 0$. Taking $k = 0$, we find again the ordinary logarithm; but there are also an infinite number of complex values for the logarithm, of the form $\log x + 2k\pi i$. If z is real and negative, we can take $\omega = \pi$; hence all the determinations of the logarithm are imaginary.

Let z' be another imaginary quantity with the absolute value ρ' and the angle ω'. We have

$$\mathrm{Log}\,(z') = \log\rho' + i(\omega' + 2k'\pi).$$

Adding the two logarithms, we obtain

$$\mathrm{Log}\,(z) + \mathrm{Log}\,(z') = \log\rho\rho' + i[\omega + \omega' + 2(k+k')\pi].$$

Since $\rho\rho'$ is equal to the absolute value of zz', and $\omega + \omega'$ is equal to its angle, this formula can be written in the form

$$\mathrm{Log}\,(z) + \mathrm{Log}\,(z') = \mathrm{Log}\,(zz'),$$

which shows that, when we add any one whatever of the values of $\mathrm{Log}\,(z)$ to any one whatever of the values of $\mathrm{Log}\,(z')$, the sum is one of the determinations of $\mathrm{Log}\,(zz')$.

Let us suppose now that the variable z describes in its plane any continuous curve whatever not passing through the origin; along this curve ρ and ω vary continuously, and the same thing is true of the different determinations of the logarithm. But two quite distinct cases may present themselves when the variable z traces a closed curve. When z starts from a point z_0 and returns to that point after having described a closed curve not containing the origin within it, the angle ω of z takes on again its original value ω_0, [[29]] and the different determinations of the logarithm come back to their initial values. If we represent each value of the logarithm by a point, each of these points traces out a closed curve. On the contrary, if the variable z describes a closed curve such as the curve M_0NMP (Fig. 3b), the angle increases by 2π, and each determination of the logarithm returns to its initial value increased by $2\pi i$. In general, when z describes any closed curve whatever, the final value of the logarithm is equal to its initial value increased by $2k\pi i$, where k denotes a positive or negative integer which gives the number of revolutions and the direction through which the radius vector joining the origin to the point z has turned. It is, then, impossible to consider the different determinations of $\operatorname{Log}(z)$ as so many distinct functions of z if we do not place any restriction on the variation of that variable, since we can pass continuously from one to the other. They are so many branches of the same function, which are permuted among themselves about the critical point $z = 0$. [[29]]

In the interior of a region which is bounded by a single closed curve and which does not contain the origin, each of the determinations of $\operatorname{Log}(z)$ is a continuous single-valued function of z. To show that it is an analytic function it is sufficient to show that it possesses a unique derivative at each point. Let z and z_1 be two neighboring values of the variable, and $\operatorname{Log}(z)$, $\operatorname{Log}(z_1)$ the corresponding values of the chosen determination of the logarithm. When z_1 approaches z, the absolute value of $\operatorname{Log}(z_1) - \operatorname{Log}(z)$ approaches zero. Let us put $\operatorname{Log}(z) = u$, $\operatorname{Log}(z_1) = u_1$; then

$$\frac{\operatorname{Log}(z_1) - \operatorname{Log}(z)}{z_1 - z} = \frac{u_1 - u}{e^{u_1} - e^u}.$$

When u_1 approaches u, the quotient

$$\frac{e^{u_1} - e^u}{u_1 - u}$$

approaches as its limit the derivative of e^u; that is, e^u or z. Hence the logarithm has a uniquely determined derivative at each point, and that derivative is equal to $1/z$. In general, $\operatorname{Log}(z - a)$ has an infinite number of determinations which permute themselves about the critical point $z = a$, and its derivative is $1/(z - a)$.

The function z^m, where m is any number whatever, real or complex, is defined by means of the equality

$$z^m = e^{m \operatorname{Log}(z)}.$$

[[30]] Unless m be a real rational number, this function possesses, just as does the logarithm, an infinite number of determinations, which permute themselves when the variable turns about the point $z = 0$. It is sufficient to make an infinite cut along a ray from the origin in order to make each branch an analytic function in the whole plane. [[30]]

The derivative is given by the expression

$$\frac{m}{z} e^{m \operatorname{Log}(z)} = m z^{m-1},$$

and it is clear that we ought to take the same value for the angle of z in the function and in its derivative.

14. Inverse functions: arc sin z, arc tan z. The inverse functions of $\sin z$, $\cos z$, $\tan z$ are defined in a similar way. Thus, the function $u = \arcsin z$ is defined by the equation

$$z = \sin u.$$

In order to solve this equation for u, we write

$$z = \frac{e^{ui} - e^{-ui}}{2i} = \frac{e^{2ui} - 1}{2ie^{ui}},$$

and we are led to an equation of the second degree,

$$U^2 - 2izU - 1 = 0, \tag{22}$$

to determine the auxiliary unknown quantity $U = e^{ui}$. We obtain from this equation

$$U = iz \pm \sqrt{1 - z^2}, \tag{23}$$

or

$$u = \text{arc sin}\, z = \frac{1}{i}\text{Log}\,(iz \pm \sqrt{1-z^2}). \tag{24}$$

The equation $z = \sin u$ has therefore two sequences of roots, which arise, on the one hand, from the two values of the radical $\sqrt{1-z^2}$, and, on the other hand, from the infinite number of determinations of the logarithm. But if one of these determinations is known, all the others can easily be determined from it. Let $U' = \rho' e^{i\omega'}$ and $U'' = \rho'' e^{i\omega'''}$ by the two roots of the equation (22); between these two roots exists the relation $U'U'' = -1$, and therefore $\rho'\rho'' = 1$, $\omega' + \omega'' = (2n+1)\pi$. It is clear that we may suppose $\omega'' = \pi - \omega'$, and we have then

$$\text{Log}\,(U') = \log \rho' + i(\omega' + 2k'\pi),$$
$$\text{Log}\,(U'') = -\log \rho' + i(\pi - \omega' + 2k''\pi).$$

[[31]] Hence all the determinations of arc sin z are given by the two formulæ [[31]]

$$\text{arc sin}\, z = \omega' + 2k'\pi - i \log \rho',$$
$$\text{arc sin}\, z = \pi + 2k''\pi - \omega' + i \log \rho',$$

and we may write

$$\text{arc sin}\, z = u' + 2k'\pi, \tag{A}$$
$$\text{arc sin}\, z = (2k''+1)\pi - u', \tag{B}$$

where $u' = \omega' - i \log \rho'$.

When the variable z describes a continuous curve, the various determinations of the logarithm in the formula (24) vary in general in a continuous manner. The only critical points that are possible are the points $z = \pm 1$, around which the two values of the radical $\sqrt{1-z^2}$ are permuted; there cannot be a value of z that causes $iz \pm \sqrt{1-z^2}$ to vanish, for, if there were, on squaring the two sides of the equation $iz = \pm\sqrt{1-z^2}$ we should obtain $1 = 0$.

Let us suppose that two cuts are made along the axis of reals, one going from $-\infty$ to the point -1, the other from the point $+1$ to $+\infty$. If the path described by the variable is not allowed to cross these cuts, the different determinations of arc sin z are single-valued functions of z. In fact, when the variable z describes a closed curve not crossing any of these cuts, the two roots U', U'' of equation (22) also describe closed curves. None of these curves contains the origin in its interior. If, for example, the curve described by the root U' contained the origin in its interior, it would cut the axis Oy in a point above Ox at least once. Corresponding to a value of U of the form $i\alpha$ ($\alpha > 0$), the relation (22) determines a value $(1+\alpha^2)/2\alpha$ for z, and this value is real and > 1. The curve described by the point z would therefore have to cross the cut which goes from $+1$ to $+\infty$.

The different determinations of arc sin z are, moreover, analytic functions of z.* For let u and u_1 be two
[[32]] neighboring values of arc sin z, corresponding to two neighboring values z and z_1 of the variable. We have [[32]]

$$\frac{u_1 - u}{z_1 - z} = \frac{u_1 - u}{\sin u_1 - \sin u}.$$

* If we choose in $U = iz + \sqrt{1-z^2}$ the determination of the radical which reduces to 1 when $z = 0$, the real part of U remains positive when the variable z does not cross the cuts, and we can put $U = Re^{i\Phi}$, where Φ lies between $-\pi/2$ and $+\pi/2$. The corresponding value of $(1/i)\text{Log}\, U$, namely,

$$\text{arc sin}\, z = \frac{1}{i}\text{Log}\, U = \Phi - i\text{Log}\, R,$$

is sometimes called the *principal value* of arc sin z. It reduces to the ordinary determination when z is real and lies between -1 and $+1$.

When the absolute value of $u_1 - u$ approaches zero, the preceding quotient has for its limit

$$\frac{1}{\cos u} = \frac{\pm 1}{\sqrt{1-z^2}}.$$

The two values of the derivative correspond to the two sequences of values (A) and (B) of arc $\sin z$.

If we do not impose any restriction on the variation of z, we can pass from a given initial value of arc $\sin z$ to any one of the determinations whatever, by causing the variable z to describe a suitable closed curve. In fact, we see first that when z describes about the point $z = 1$ a closed curve to which the point $z = -1$ is exterior, the two values of the radical $\sqrt{1-z^2}$ are permuted and so we pass from a determination of the sequence (A) to one of the sequence (B). Suppose next that we cause z to describe a circle of radius R ($R > 1$) about the origin as center; then each of the two points U', U'' describes a closed curve. To the point $z = +R$ the equation (22) assigns two values of U, $U' = i\alpha$, $U'' = i\beta$, where α and β are positive; to the point $z = -R$ there correspond by means of the same equation the values $U' = -i\alpha'$, $U'' = -i\beta'$, where α' and β' are again positive. hence the closed curves described by these two points U', u'' cut the axis Oy in two points, one above and the other below the point O; each of the logarithms $\operatorname{Log}(U')$, $\operatorname{Log}(U'')$ increases or diminishes by $2\pi i$.

In the same way the function arc $\tan z$ is defined by means of the relation $\tan u = z$, or

$$z = \frac{1}{i}\frac{e^{2ui}-1}{e^{2ui}+1};$$

whence we have

$$e^{2ui} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z},$$

and consequently

$$\text{arc}\tan z = \frac{1}{2i}\operatorname{Log}\left(\frac{i-z}{i+z}\right).$$

This expression shows the two logarithmic critical points $\pm i$ of the function arc $\tan z$. When the variable z passes around one of these points, $\operatorname{Log}[(i-z)/(i+z)]$ increases or diminishes by $2\pi i$, and arc $\tan z$ increases or diminishes by π.

[[33]] **15. Applications to the integral calculus.** The derivatives of the functions which we have just defined have the same form as when the variable is real. Conversely, the rules for finding primitive functions apply also to the elementary functions of complex variables. Thus, denoting by $\int f(z)\,dz$ a function of the complex variable z whose derivative is $f(z)$, we have [[33]]

$$\int \frac{A\,dz}{(z-a)^m} = -\frac{A}{m-1}\frac{1}{(z-a)^{m-1}}, \qquad (m > 1),$$

$$\int \frac{A\,dz}{z-a} = A\operatorname{Log}(z-a).$$

These two formulæ enable us to find a primitive function of any rational function whatever, with real or imaginary coefficients, provided the roots of the denominator are known. Consider as a special case a rational function of the real variable x with real coefficients. If the denominator has imaginary roots, they occur in conjugate pairs, and each root has the same multiplicity as its conjugate. Let $\alpha + \beta i$ and $\alpha - \beta i$ be two conjugate roots of multiplicity p. In the decomposition into simple fractions, if we proceed with the imaginary roots just as with the real roots, the root $\alpha + \beta i$ will furnish a sum of simple fractions

$$\frac{M_1 + N_1 i}{x-\alpha-\beta i} + \frac{M_2 + N_2 i}{(x-\alpha-\beta i)^2} + \cdots + \frac{M_p + N_p i}{(x-\alpha-\beta i)^p},$$

and the root $\alpha - \beta i$ will furnish a similar sum, but with numerators that are conjugates of the former ones. Combining in the primitive function the terms which come from the corresponding fractions, we shall have, if $p > 1$,

$$\int \frac{M_p + N_p i}{(x-\alpha-\beta i)^p}\,dx + \int \frac{M_p - N_p i}{x-\alpha+\beta i)^p}\,dx = -\frac{1}{p-1}\left[\frac{M_p + N_p i}{(x-\alpha-\beta i)^{p-1}} + \frac{M_p - N_p i}{(x-\alpha+\beta i)^{p-1}}\right]$$

$$= -\frac{1}{p-1}\frac{(M_p + N_p i)(x-\alpha+\beta i)^{p-1} + \cdots}{[(x-\alpha)^2+\beta^2]^{p-1}},$$

and the numerator is evidently the sum of two conjugate imaginary polynomials. If $p = 1$, we have

$$\int \frac{M_1 + N_1 i}{x - \alpha - \beta i}\,dx + \int \frac{M_1 - N_1 i}{x - \alpha + \beta i}\,dx = (M_1 + N_1 i)\mathrm{Log}\,[(x-\alpha) - \beta i] + (M_1 - N_1 i)\mathrm{Log}\,[(x-\alpha) + \beta i].$$

[[34]] If we replace the logarithms by their developed expressions, there remains on the right-hand side [[34]]

$$M_1 \log[(x-\alpha)^2 + \beta^2] + 2N_1 \mathrm{arc\ tan}\,\frac{\beta}{x-\alpha}.$$

It suffices to replace

$$\mathrm{arc\ tan}\,\frac{\beta}{x-\alpha} \quad \text{by} \quad \frac{\pi}{2} - \mathrm{arc\ tan}\,\frac{x-\alpha}{\beta}$$

in order to express the result in the form in which it is obtained when imaginary symbols are not used.

Again, consider the indefinite integral

$$\int \frac{dx}{\sqrt{Ax^2 + 2Bx + C}},$$

which has two essentially different forms, according to the sign of A. The introduction of complex variables reduces the two forms to a single one. In fact, if in the formula

$$\int \frac{dx}{\sqrt{1+x^2}} = \mathrm{Log}\left(x + \sqrt{1+x^2}\right)$$

we change x to ix, there results

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{1}{i}\mathrm{Log}\left(ix + \sqrt{1-x^2}\right),$$

and the right-hand side represents precisely arc $\sin x$.

The introduction of imaginary symbols in the integral calculus enables us, then, to reduce one formula to another even when the relationship between them might not be at all apparent if we were to remain always in the domain of real numbers.

We shall give another example of the simplification which comes from the use of imaginaries. If a and b are real, we have

$$\int e^{(a+bi)x}\,dx = \frac{e^{(a+bi)x}}{a+bi} = \frac{a-bi}{a^2+b^2}e^{ax}(\cos bx + i\sin bx).$$

Equating the real parts and the coefficients of i, we have at one stroke two integrals already calculated (I, §109, 2d ed.; §119, 1st ed.):

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2},$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$$

[[35]] In the same way we may reduce the integrals [[35]]

$$\int x^m e^{ax}\cos bx\,dx, \qquad \int x^m e^{ax}\sin bx\,dx$$

to the integral $\int x^m e^{(a+bi)x}\,dx$, which can be calculated by a succession of integrations by parts, where m is any integer.

16. Decomposition of a rational function of sin z and cos z into simple elements. Given a rational function of $\sin z$ and $\cos z$, $F(\sin z, \cos z)$, if in it we replace $\sin z$ and $\cos z$ by their expressions given by Euler's formula, it becomes a rational function $R(t)$ of $t = e^{zi}$. This function $R(t)$, decomposed into simple elements, will be made up of an integral part and a sum of fractions coming from the roots of the denominator of $R(t)$. If that denominator has the root $t = 0$, we shall combine with the integral part the fractions arising from that root, which will give a polynomial or a rational function $R_1(t) = \sum K_m t^m$, where the exponent m may have negative values.

Let $t = a$ be a root of the denominator different from zero. That root will give rise to a sum of simple fractions

$$f(t) = \frac{A_1}{t-a} + \frac{A_2}{(t-a)^2} + \cdots + \frac{A_n}{(t-a)^n}.$$

The root a not being zero, let α be a root of the equation $e^{\alpha i} = a$; then $1/(t-a)$ can be expressed very simply by means of $\operatorname{ctn}[(z-\alpha)/2]$. We have, in fact,

$$\operatorname{ctn}\frac{z-\alpha}{2} = i\frac{e^{zi}+e^{\alpha i}}{e^{zi}-e^{\alpha i}} = i\left(1+\frac{2e^{\alpha i}}{e^{zi}-e^{\alpha i}}\right);$$

whence it follows that

$$\frac{1}{t-a} = \frac{1}{e^{zi}-e^{\alpha i}} = -\frac{1}{2e^{\alpha i}}\left(1+i\operatorname{ctn}\frac{z-\alpha}{2}\right).$$

Hence the rational fraction $f(t)$ changes to a polynomial of degree n in $\operatorname{ctn}[(z-\alpha)/2]$,

$$A_0' + A_1'\operatorname{ctn}\frac{z-\alpha}{2} + A_2'\operatorname{ctn}^2\left(\frac{z-\alpha}{2}\right) + \cdots + A_n'\operatorname{ctn}^n\left(\frac{z-\alpha}{2}\right).$$

The successive powers of the cotangent up to the nth can be expressed in turn in terms of its successive derivatives up to the $(n-1)$th; we have first

$$\frac{d\operatorname{ctn} z}{dz} = -\frac{1}{\sin^2 z} = -1 - \operatorname{ctn}^2 z,$$

[[36]] which enables us to express $\operatorname{ctn}^2 z$ in terms of $d(\operatorname{ctn} z)/dz$, and it is easy to show, by mathematical induction, that if the law is true up to $\operatorname{ctn}^n z$, it will also be true for $\operatorname{ctn}^{n+1} z$. The preceding polynomial of degree n in $\operatorname{ctn}[(z-\alpha)/2]$ will change to a linear expression in $\operatorname{ctn}[(z-\alpha_/2]$ and its derivatives, [[36]]

$$\mathcal{A}_0 + \mathcal{A}_1\operatorname{ctn}\frac{z-\alpha}{2} + \mathcal{A}_2\frac{d}{dz}\left(\operatorname{ctn}\frac{z-\alpha}{2}\right) + \cdots + \mathcal{A}_n\frac{d^{n-1}}{dz^{n-1}}\left(\operatorname{ctn}\frac{z-\alpha}{2}\right).$$

Let us proceed in the same way with all the roots $b, c, \cdots, l$ of the denominator of $R(t)$ different from zero, and let us add the results obtained after having replaced t by e^{zi} in $R_1(t)$. The given rational function $F(\sin z, \cos z)$ will be composed of two parts,

$$F(\sin z, \cos z) = \Phi(z) + \Psi(z). \tag{25}$$

The function $\Phi(z)$, which corresponds to the integral part of a rational function of the variable, is of the form

$$\Phi(z) = C + \sum(\alpha_m \cos mz + \beta_m \sin mz), \tag{26}$$

where m is an integer not zero. On the other hand, $\Psi(z)$, which corresponds to the fractional part of a rational function, is an expression of the form

$$\begin{aligned}\Psi(z) = {} & \mathcal{A}_1\operatorname{ctn}\left(\frac{z-\alpha}{2}\right) + \mathcal{A}_2\frac{d}{dz}\operatorname{ctn}\left(\frac{z-\alpha}{2}\right) + \cdots + \mathcal{A}_n\frac{d^{n-1}}{dz^{n-1}}\operatorname{ctn}\left(\frac{z-\alpha}{2}\right)\\ & + \mathcal{B}_1\operatorname{ctn}\left(\frac{z-\beta}{2}\right) + \mathcal{B}_2\frac{d}{dz}\operatorname{ctn}\left(\frac{z-\beta}{2}\right) + \cdots + \mathcal{B}_p\frac{d^{p-1}}{dz^{p-1}}\operatorname{ctn}\left(\frac{z-\beta}{2}\right)\\ & + \cdots\end{aligned}$$

It is the function ctn $[(z-\alpha)/2]$ which here plays the rôle of the simple element, just as the fraction $1/(z-\alpha)$ does for a rational function. The result of this decomposition of $F(\sin z, \cos z)$ is easily integrated; we have, in fact,

$$\int \operatorname{ctn}\left(\frac{z-\alpha}{2}\right) dz = 2\mathrm{Log}\left[\sin\left(\frac{z-\alpha}{2}\right)\right], \tag{27}$$

and the other terms are integrable at once. In order that the primitive function may be periodic, it is necessary and sufficient that all the coefficients $C, \mathcal{A}_1, \mathcal{B}_1, \cdots$ be zero.

In practice it is not always necessary to go through all these successive transformations in order to put the function $F(\sin z, \cos z)$ into its final form (25). Let α be a value of z which makes the function F infinite.
[[37]] We can always calculate, by a simple division, the coefficients of $(z-\alpha)^{-1}, (z-\alpha)^{-2}, \cdots$, in the part that [[37]]
is infinite for $z = \alpha$ (I, §188, 2d ed.; §183, 1st ed.). On the other hand, we have

$$\operatorname{ctn}\left(\frac{z-\alpha}{2}\right) = \frac{2}{z-\alpha} + P(z-\alpha),$$

where $P(z-\alpha)$ is a power series; equating the coefficients of the successive powers of $(z-\alpha)^{-1}$ in the two sides of the equation (25), we shall then obtain easily $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$.

Consider, for example, the function $1/(\cos z - \cos\alpha)$. Setting $e^{zi} = t$, $e^{\alpha i} = a$, it takes the form

$$\frac{2at}{a(t^2+1) - t(a^2+1)}.$$

The denominator has the two simple roots $t = a, t = 1/a$, and the numerator is of lower degree than the denominator. We shall have, then, a decomposition of the form

$$\frac{1}{\cos z - \cos\alpha} = C + \mathcal{A}\operatorname{ctn}\frac{z-\alpha}{2} + \mathcal{B}\operatorname{ctn} z + \alpha 2.$$

In order to determine $\mathcal{A}$, let us multiply the two sides by $z-\alpha$, and let us then put $z = \alpha$. This gives $\mathcal{A} = -1/(2\sin\alpha)$. In a similar manner, we find $\mathcal{B} = 1/(2\sin\alpha)$. Replacing $\mathcal{A}$ and $\mathcal{B}$ by these values and setting $z = 0$, it is seen that $C = 0$, and the formula takes the form

$$\frac{1}{\cos z - \cos\alpha} = \frac{1}{2\sin\alpha}\left(\operatorname{ctn}\frac{z+\alpha}{2} - \operatorname{ctn}\frac{z-\alpha}{2}\right).$$

Let us now apply the general method to the integral powers of $\sin z$ and of $\cos z$. We have, for example,

$$(\cos z)^m = \left(\frac{e^{zi}+e^{-zi}}{2}\right)^m.$$

Combining the terms at equal distances from the extremities of the expansion of the numerator, and then applying Euler's formulæ, we find at once

$$(2\cos z)^m = 2\cos mz + 2m\cos(m-2)z + 2\frac{m(m-1)}{1.2}\cos(m-4)z + \cdots.$$

If m is odd, the last term contains $\cos z$; if m is even, the term which ends the expansion is independent of z and is equal to $m!/[(m/2)!]^2$. In the same way, if m is odd,

$$(2i\sin z)^m = 2i\sin mz - 2imsin(m-2)z + 2i\frac{m(m-1)}{1.2}\sin(m-4)z\cdots;$$

and if m is even,

$$(2i\sin z)^m = 2\cos mz - 2m\cos(m-2)z + \cdots + (-1)^{\frac{m}{2}}\frac{m!}{\left(\frac{m}{2}!\right)^2}.$$

These formulæ show at once that the primitive functions of $(\sin z)^m$ and of $(\cos z)^m$ are periodic functions of z when m is *odd*, and only then.

[[38]] *Note.* When the function $F(\sin z, \cos z)$ has the period π, we can express it rationally in terms of e^{2zi} [[38]]
and can take for the simple elements $\operatorname{ctn}(z-\alpha)$, $\operatorname{ctn}(z-\beta)$, $\cdots$.

17. Expansions of Log $(1+z)$**.** The transcendental functions which we have defined are of two kinds: those which, like e^z, $\sin z$, $\cos z$, are analytic in the whole plane, and those which, like $\operatorname{Log} z$, $\operatorname{arc\,tan} z$, $\cdots$, have singular points and cannot be represented by developments in power series convergent in the whole plane. Nevertheless, such functions may have developments holding for certain parts of the plane. We shall now show this for the logarithmic function.

Simple division leads to the elementary formula

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n \pm z^{n+1} 1 + z;$$

and if $|z| < 1$, the remainder $z^{n+1}/(1+z)$ approaches zero when n increases indefinitely. Hence, in the interior of a circle C of radius 1 we have

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n \pm \cdots.$$

Let $F(z)$ be the series obtained by integrating this series term by term:

$$F(z) = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots + (-1)^n \frac{z^{n+1}}{n+1} + \cdots;$$

this new series is convergent inside the unit circle and represents an analytic function whose derivative $F'(z)$ is $1/(1+z)$. We know, however, a function which has the same derivative, $\operatorname{Log}(1+z)$. It follows that the difference $\operatorname{Log}(1+z) - F(z)$ reduces to a constant.* In order to determine this constant it will be necessary to fix precisely the determination chosen for the logarithm. If we take the one which becomes zero for $z = 0$, we have for every point inside C

$$\operatorname{Log}(1+z) = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots. \tag{28}$$

Let us join the point A to the point M, which represents z (Fig. 8). The absolute value of $1+z$ is represented

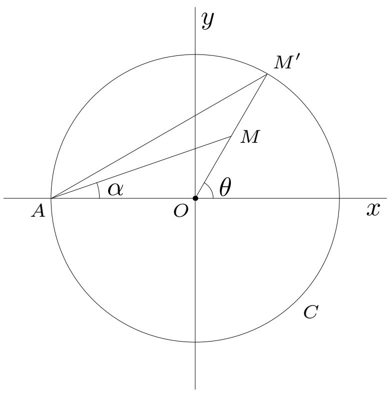

FIG. 8

by the length $r = AM$. For the angle of $1+z$ we can take the angle α which AM makes with AO, an angle which lies between $-\pi/2$ and $+\pi/2$ as long as the point M remains inside the circle C. That determination
[[39]] of the logarithm which becomes zero for $z = 0$ is $\log r + i\alpha$; hence the formula (28) is not ambiguous. [[39]]

* In order that the derivative of an analytic function $X + Yi$ be zero, it is necessary that we have (§3) $\partial X/\partial x = 0$, $\partial Y \partial x = 0$, and consequently $\partial Y/\partial y = \partial X/\partial y = 0$; X and Y are therefore constants.

Changing z to $-z$ in this formula and then subtracting the two expressions, we obtain

$$\operatorname{Log}\left(\frac{1+z}{1-z}\right) = 2\left(\frac{z}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right).$$

If we now replace z by iz, we shall obtain again the development of arc tan z

$$\text{arc tan}\, z = \frac{1}{2i}\operatorname{Log}\left(\frac{1+iz}{1-iz}\right) = \frac{z}{1} - \frac{z^3}{3} + \frac{z^5}{5} - \cdots.$$

The series (28) remains convergent at every point on the circle of convergence except the point A (footnote, p. 19), and consequently the two series

$$\cos\theta - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \cos 4\theta 4 + \cdots,$$
$$\sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \cdots$$

are both convergent except for $\theta = (2k+1)\pi$ (cf. I, §166). By Abel's theorem the sum of the series at M' is the limit approached by the sum of the series at a point M as M approaches M' along the radius OM'. If we suppose θ always between $-\pi$ and $+\pi$, the angle α will have fot its limit $\theta/2$, and the absolute value AM will behave for its limit $2\cos(\theta/2)$. We can therefore write

$$\log\left(2\cos\frac{\theta}{2}\right) = \cos\theta - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \frac{\cos 4\theta}{4} + \cdots,$$
$$\frac{\theta}{2} = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \cdots, \qquad (-\pi < \theta < \pi).$$

If in the last formula we replace θ by $\theta - \pi$, we obtain again a formula previously established (I, §204, 2d ed.; §198, 1st ed.).

[[40]] **18. Extension of the binomial formula.** In a fundamental paper on power series, Abel set for himself the problem of determining the sum of the convergent series [[40]]

$$(29)\qquad \begin{cases} \phi(m,z) = 1 + \dfrac{m}{1}z + \dfrac{m(m-1)}{1.2}z^2 + \cdots \\ \qquad + \dfrac{m(m-1)\cdots(m-p+1)}{p!}z^p + \cdots \end{cases}$$

for all the values of m and z, real or imaginary, provided we have $|z| < 1$. We might accomplish this by means of a differential equation, in the manner indicated in the case of real variables (I, §183, 2d ed.; §179, 1st ed.). The following method, which gives an application of §11, is more closely related to the method followed by Abel. We shall suppose z fixed and $|z| < 1$, and we shall study the properties of $\phi(m,z)$ considered as a function of m. If m is a positive integer, the function evidently reduces to the polynomial $(1+z)^m$. *If m and m' are any two values whatever of the parameter m, we have always*

$$(30)\qquad \phi(m,z)\phi(m',z) = \phi(m+m',z).$$

In fact, let us multiply the two series $\phi(m,z)$, $\phi(m',z)$ by the ordinary rule. The coefficient of z^p in the product is equal to

$$(31)\qquad m_p + m_{p-1}m'_1 + m_{p-2}m'_2 + \cdots + m_1 m'_{p-1} + m'_p,$$

where we have set for abbreviation

$$m_k = \frac{m(m-1)\cdots(m-k+1)}{k!}.$$

The proposed functional relation will be established if we show that the expression (31) is identical with the coefficient of z^p in $\phi(m+m',z)$, that is, with $(m+m')_p$. We could easily verify directly the identity

$$(32)\qquad (m+m')_p = m_p + m_{p-1}m'_1 + \cdots + m'_p,$$

but the computation is unnecessary if we notice that the relation (30) is always satisfied whenever m and m' are positive integers. The two sides of the equation (32) are polynomials in m and m' which are equal whenever m and m' are positive integers; they are therefore identical.

On the other hand, $\phi(m,z)$ can be expanded in a power series of increasing powers of m. In fact, if we
[[41]] carry out the indicated products, $\phi(m,z)$ can be considered as the sum of a double series [[41]]

$$(33)\qquad \begin{cases} \phi(m,z) = 1 + \dfrac{m}{1}z - \dfrac{m}{2}z^2 + \dfrac{m}{3}z^3 - \cdots \pm \dfrac{m}{p}z^p \mp \cdots \\ \qquad\qquad + \dfrac{m^2}{2}z^2 - \dfrac{m^2}{2}z^3 + \cdots \\ \qquad\qquad\qquad + \dfrac{m^3}{6}z^3 - \cdots + \dfrac{m^p}{p!}z^p + \cdots \end{cases}$$

if we sum it by columns. This double series is absolutely convergent. For, let $|z| = \rho$ and $|m| = \sigma$; if we replace each term by its absolute value, the sum of the terms of the new series included in the $(p+1)$th column is equal to

$$\frac{\sigma(\sigma+1)\cdots(\sigma+p-1)}{p!}\rho^p,$$

which is the general term of a convergent sires. We can therefore sum the double series by rows, and we thus obtain for $\phi(m,z)$ a development in power series

$$\phi(m,z) = 1 + \frac{a_1}{1}m + \frac{a_2}{1.2}m^2 + \cdots.$$

From the relation (30) and the results established above (§11), this series must be identical with that for $e^{a_1 m}$. Now for the coefficient of m we have

$$a_1 = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \mathrm{Log}\,(1+z);$$

hence

$$(34)\qquad \phi(m,z) = e^{m\,\mathrm{Log}\,(1+z)},$$

where the determination of the logarithm to be understood is that one which becomes zero when $z=0$. We can again represent the last expression by $(1+z)^m$; but in order to know without ambiguity the value in question, it is convenient to make use of the expression $e^{m\,\mathrm{Log}\,(1+z)}$.

Let $m = \mu + \nu i$; if r and α have the same meanings as in the preceding paragraph, we have

$$\begin{aligned} e^{m\,\mathrm{Log}\,(1+z)} &= e^{(\mu+\nu i)(\log r + i\alpha)} \\ &= e^{\mu\log r - \nu\alpha}\left[\cos(\mu\alpha + \nu\log r) + i\sin(\mu\alpha+\nu\log r)\right]. \end{aligned}$$

In conclusion, let us study the series on the circle of convergence. Let U_n be the absolute value of the general term for a point z on the circle. The ratio of two consecutive terms of the series of absolute values is equal to $|(m-n+1)/n|$, that is, if $m = \mu+\nu i$, to

$$\frac{\sqrt{(\mu+1-n)^2+\nu^2}}{n} = 1 - \frac{\mu+1}{n} + \frac{\phi(n)}{n^2},$$

[[42]] where the function $\phi(n)$ remains finite when n increases indefinitely. By a known rule for convergence (I, §163) this series is convergent when $\mu+1>1$ and divergent in every other case. *The series* (29) *is therefore absolutely convergent at all the points on the circle of convergence when μ is positive.* [[42]]

If $\mu+1$ is negative or zero, the absolute value of the general term never decreases, since the ratio U_{n+1}/U_n is never less than unity. *The series is divergent at all the points on the circle when $\mu \le -1$.*

It remains to study the case where $-1 < \mu \le 0$. Let us consider the series whose general term is U_n^p; the ratio of two consecutive terms is equal to

$$\left[1 - \frac{\mu+1}{n} + \frac{\phi(n)}{n^2}\right]^p = 1 - \frac{p(\mu+1)}{n} + \frac{\phi_1(n)}{n^2},$$

and if we choose p large enough so that $p(\mu+1) > 1$, this series will be convergent. It follows that U_n^p, and consequently the absolute value of the general term U_n, approaches zero. This being the case, in the identity

$$\phi(m, z)(1+z) = \phi(m+1), z)$$

let us retain on each side only the terms of degree less than or equal to n; there remains the relation

$$S_n(1+z) = S'_n + \frac{m(m-1)\cdots(m-n+1)}{n!} z^{n+1},$$

where S_n and S'_n indicate respectively the sum of the first $(n+1)$ terms of $\phi(m,z)$ and of $\phi(m+1,z)$. If the real part of m lie between -1 and 0, the real part of $m+1$ is positive. Suppose $|z| = 1$; when the number n increases indefinitely, S'_n approaches a limit, and the last term on the right approaches zero; it follows that S_n also approaches a limit, unless $1+z = 0$. Therefore, *when* $-1 < \mu \leq 0$, *the series is convergent at all the points on the circle of convergence, except at the point* $z = -1$.

III. CONFORMAL REPRESENTATION

19. Geometric interpretation of the derivative. Let $u = X + Yi$ be a function of the complex variable z, analytic within a closed curve C. We shall represent the value of u by the point whose coördinates are X, Y with respect to a system of rectangular axes. To simplify the following statements we shall suppose that the axes OX, OY are parallel respectively to the axes Ox and Oy and arranged in the same order of rotation in the same plane or in a plane parallel to the plane xOy.

When the point z describes the region A bounded by the closed curve C, the point u with the coördinates (X, Y) describes in its plane a region A'; the relation $u = f(z)$ defines then a certain correspondence between the points of the two planes or of two portions of a plane. On account of the relations which connect the derivatives of the functions X, y, it is clear that this correspondence should possess special properties. We shall now show that *the angles are unchanged.*

[[43]] Let z and z_1 be two neighboring points of the region A, and u and u_1 the corresponding points of the region A'. By the original definition of the derivative the quotient $(u_1 - u)/(z_1 - z)$ has for its limit $f'(z)$ when the absolute value of $z_1 - z$ approaches zero in any manner whatever. Suppose that the point z_1 approaches the point z along a curve C, whose tangent at the point z makes an angle α with the parallel to the direction Ox; the point u_1 will itself describe a curve C' passing through u. Let us discard the case in which $f'(z)$ is zero, and let ρ and ω be the absolute value and the angle of $f'(z)$ respectively. Likewise let r and r' be the distances zz_1 and uu_1, α' the angle which the direction zz_1 makes with the parallel zx' to Ox, and β' the angle which the direction uu_1 makes with the parallel uX' to OX. The absolute value of the quotient $(u_1 - u)/(z_1 - z)$ is equal to r_1/r, and the angle of the quotient is equal to $\beta' - \alpha'$. We have then

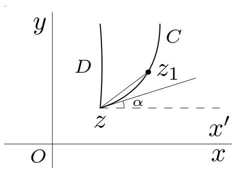

FIG. 9 *a*

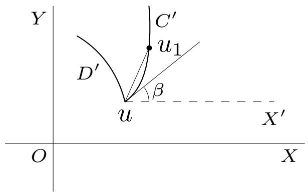

FIG. 9 *b*

the two relations

$$\lim \frac{r_1}{r} = \rho, \qquad \lim(\beta' - \alpha') = \omega + 2k\pi. \tag{35}$$

Let us consider only the second of these relations. We may suppose $k = 0$, since a change in k simply causes an increase in the angle ω by a multiple of 2π. When the point z_1 approaches the point z along the curve C, α' approaches the limit α, β' approaches a limit β, and we have $\beta = \alpha + \omega$. That is to say, *in order to obtain the direction of the tangent to the curve described by the point* u, *it suffices to turn the direction of the tangent to the curve described by* z *through a constant angle* ω. It is naturally understood in this statement that those directions of the two tangents are made to correspond which correspond to the same sense of motion of the points z and u.

Let D be another curve of the plane xOy passing through the point z, and let D' be the corresponding curve of the plane XOY. If the letters γ and δ denote respectively the angles which the corresponding [[44]] directions of the tangents to these two curves make with zx' and uX' (Figs. 9 *a* and 9 *b*), we have

$$\beta = \alpha + \omega, \qquad \delta = \gamma + \omega,$$

and consequently $\delta - \beta = \gamma - \alpha$. *The curves C' and D' cut each other in the same angle as the curves C and D.* Moreover, we see that the sense of rotation is preserved. It should be noticed that if $f'(z) = 0$, the demonstration no longer applies.

If, in particular, we consider, in one of the two planes xOy or XOY, two families of orthogonal curves, the corresponding curves in the other plane also will form two families of orthogonal curves. For example, the two families of curves $X = C$, $Y = C'$, and the two families of curves

$$|f(z)| = C, \qquad \text{angle}\, f(z) = C' \tag{36}$$

form orthogonal nets in the plane xOy, for the corresponding curves in the plane XOY are, in the first case, two systems of parallels to the axes of coördinates, and, in the other, circles having the origin for center and straight lines proceeding from the origin.

Example 1. Let $z' = z^\alpha$, where α is a real positive number. Indicating by r and θ the polar coördinates of z, and by r' and θ' the polar coördinates of z', the preceding relation becomes equivalent to the two relations $r' = r^\alpha$, $\theta' = \alpha\theta$. We pass then from the point z to the point z' by raising the radius vector to the power α and by multiplying the angle by α. The angles are preserved, except those which have their vertices at the origin, and these are multiplied by the constant factor α.

Example 2. Let us consider the general linear transformation

$$z' = \frac{az + b}{cz + d}, \tag{37}$$

where a, b, c, d are any constants whatever. In certain particular cases it is easily seen how to pass from the point z to the point z'. Take for example the transformation $z' = z + b$; let $z = z + yi$, $z' = z' + y'i$, $b = \alpha + \beta i$; the preceding relation gives $x' = x + \alpha$, $y' = y + \beta$, which shows that we pass from the point z to the point z' by a translation.

Let now $z' = az$; if ρ and ω indicate the absolute value and angle of a respectively, then we have $r' = \rho r$, $\theta' + \omega + \theta$. Hence we pass from the point z to the point z' by multiplying the radius vector by the constant factor ρ and then turning this new radius vector through a constant angle ω. We obtain then the transformation defined by the formula $z' = az$ by combining an expansion with a rotation.

Finally, let us consider the relation

$$z' = \frac{1}{z},$$

where r, θ, r', θ' have the same meanings as above. We must have $rr' = 1$, $\theta + \theta' = 0$. The product of the radii vectors is [[45]] therefore equal to unity, while the polar angles are equal and of opposite signs. Given a circle C with center A and radius R, we shall use the expression *inversion with respect to the given circle* to denote the transformation by which the polar angle is unchanged but the radius vector of the new point is R^2/r. We obtain then the transformation defined by the relation $z'z = 1$ by carrying out first an inversion with respect to a circle of unit radius and with the origin as center, and then taking the symmetric point to the point obtained with respect to the axis Ox.

The most general transformation of the form (37) can be obtained by combining the transformations which we have just studied. If $c = 0$, we can replace the transformation (37) by the succession of transformations

$$z_1 = \frac{a}{d} z, \qquad z' + z_1 + \frac{b}{d}.$$

If c is *not* zero, we can carry out the indicated division and write

$$z' = \frac{a}{c} + \frac{bc - ad}{c^2 z + cd},$$

and the transformation can be replaced by the succession of transformations

$$z_1 = z + \frac{d}{c}, \qquad z_2 = c^2 z_1, \qquad z_3 = \frac{1}{z_2}$$
$$z_4 = (bc - ad) z_2, \qquad z' = z_4 + \frac{a}{c}.$$

All these special transformations leave the angles and the sense of rotation unchanged, and change circles into circles. Hence the same thing is then true of the general transformation (37), which is therefore often called a *circular transformation.* In the above statement straight lines should be regarded as circles with infinite radii.

Example 3. Let

$$z' = (z - e_1)^{m_1} (z - e_2)^{m_2} \cdots (z - e_p)^{m_p},$$

where e_1, e_2, $\cdots$, e_p are any quantities whatever, and where the exponents m_1, m_2, $\cdots$, m_p are any real numbers, positive or negative. Let M, E_1, E_2, $\cdots$, E_p be the points which represent the quantities z, e_1, e_2, $\cdots$, e_p; let also r_1, r_2, $\cdots$, r_p denote the distances ME_1, ME_2, $\cdots$, ME_p and θ_1, θ_2, $\cdots$, θ_p the angles which E_1M, E_2M, $\cdots$, E_pM make with the parallels to Ox. The absolute value and the angle of z' are respectively $r_1^{m_1} r_2^{m_2} \cdots r_p^{m_p}$ and $m_1\theta_1 + m_2\theta_2 + \cdots + m_p\theta_p$. Then the two families of curves

$$r_1^{m_1} r_2^{m_2} \cdots r_p^{m_p} = C, \qquad m_1\theta_1 + m_2\theta_2 + \cdots + m_p\theta_p = C'$$

form an orthogonal system. When the exponents m_1, m_2, $\cdots$, m_p are rational numbers, all the curves are algebraic. If, for example, $p = 2$, $m_1 = m_2 = 1$, one of the families is composed of Cassinian ovals with two foci, and the second family is a system of equilateral hyperbolas.

20. Conformal transformations in general. The examination of the converse of the proposition which we have just established leads us to treat a more general problem. Two surfaces Σ, Σ', being given, let
[[46]] us set up between them any point-to-point correspondence whatever (except for certain broad restrictions [[46]]
which will be made later), and let us examine the cases in which the angles are unaltered in that transformation. Let x, y, z be the rectangular coördinates of a point of Σ, and let x', y', z' be the rectangular coördinates of a point of Σ'. We shall suppose the six coördinates x, y, z, x', y', z' expressed as functions of two variable parameters u, v in such a way that corresponding points of the two surfaces correspond to the same pair of values of the parameters u, v:

$$(38) \qquad \Sigma \begin{cases} x = f(u,v), \\ y = \phi(u,v), \\ z = \psi(u,v), \end{cases} \qquad \Sigma' \begin{cases} x' = f'(u,v), \\ y' = \phi'(u,v), \\ z' = \psi'(u,v). \end{cases}$$

Moreover, we shall suppose that the functions f, ϕ, $\cdots$, together with their partial derivatives of the first order, are continuous when the points (x, y, z) and (x', y', z') remain in certain regions of the two surfaces Σ and Σ'. We shall employ the usual notations (I, §131):

$$(39) \qquad \begin{cases} E = S\left(\dfrac{\partial x}{\partial u}\right)^2, \qquad F = S\dfrac{\partial x}{\partial u}\dfrac{\partial x}{\partial v}, \qquad G = S\left(\dfrac{\partial x}{\partial v}\right)^2, \\ E' = S\left(\dfrac{\partial x'}{\partial u}\right)^2, \qquad F' = S\dfrac{\partial x'}{\partial u}\dfrac{\partial x'}{\partial v}, \qquad G' = S\left(\dfrac{\partial x'}{\partial v}\right)^2, \\ ds^2 = Edu^2 + 2Fdudv + Gdv^2, \\ ds'^2 = E'du^2 + 2F'dudv + G'dv^2. \end{cases}$$

Let C and D (Figs. 10 a and 10 b) be two curves on the surfaces Σ, passing through a point m of that surface, and C' and D' the corresponding curves on the surface Σ' passing through the point m'. Along

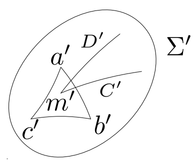

FIG. 10 a

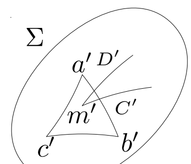

FIG. 10 b

the curve C the parameters u, v are functions of a single auxiliary variable t, and we shall indicate their differentials by du and dv. Likewise, along D, u and v are functions of a variable t', and we shall denote their differentials here by δu and δv. In general, we shall distinguish by the letters d and δ the differentials
[[47]] relative to a displacement on the curve C and to one on the curve D. The following total differentials are [[47]]
proportional to the direction cosines of the tangent to the curve C,

$$ds = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \qquad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv, \qquad dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv,$$

and the following are proportional to the direction cosines of the tangent to the curve D,

$$\delta x = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v, \qquad \delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v, \qquad \delta z = \frac{\partial z}{\partial u}\delta u + \frac{\partial z}{\partial v}\delta v.$$

Let ω be the angle between the tangents to the two curves C and D. The value of $\cos\omega$ is given by the expressions

$$\cos\omega = \frac{dx\delta x + dy\delta y + dz\delta z}{\sqrt{dx^2+dy^2+dz^2}\sqrt{\delta x^2+\delta y^2+\delta z^2}},$$

which can be written, making use of the notation (39), in the form

$$\cos\omega = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{\sqrt{Edu^2+2Fdudv+Gdv^2}\sqrt{E\delta u^2+2F\delta u\delta v+G\delta v^2}}. \tag{40}$$

If we let ω' denote the angle between the tangents to the two curves C' and D', we have also

$$\cos\omega' = \frac{E'du\delta u + F'(du\delta v + dv\delta u) + G'dv\delta v}{\sqrt{E'du^2+2F'dudv+G'd^2}\sqrt{E'\delta u^2+2F'\delta u\delta v+G'\delta v^2}}. \tag{41}$$

In order that the transformation considered shall not change the value of the angles, it is necessary that $\cos\omega' = \cos\omega$, whatever du, dv, δu, δv may be .The two sides of the equality

$$\cos^2\omega' = \cos^2\omega$$

are rational functions of the ratios $\delta v/\delta u$, dv/du, and these functions must be equal whatever the values of these ratios. Hence the corresponding coefficients of the two fractions must be proportional; that is, we must have

$$\frac{E'}{E} = \frac{F'}{F} = \frac{G'}{G} = \lambda^2, \tag{42}$$

where λ is any function whatever of the parameters u, v. These conditions are evidently also sufficient, for $\cos\omega$, for example, is a homogeneous function of E, F, G, of degree zero.

The conditions (42) can be replaced by a single relation $ds'^2 = \lambda^2 ds^2$, or

$$ds' = \lambda ds. \tag{43}$$

[[48]] This relation states that the ratio of two corresponding infinitesimal arcs approach a limit independent of du [[48]]
and dv, when these two arcs approach zero. This condition makes the reasoning almost intuitive. For, let abc be an infinitesimal triangle on the first surface, and $a'b'c'$ the corresponding triangle on the second surface. Imagine these two curvilinear triangles replaced by rectilinear triangles that approximate them. Since the ratios $a'b'/ab$, $a'c'/ac$, $b'c'/bc$ approach the same limit $\lambda(u,v)$, these two triangles approach similarity and the corresponding angles approach equality.

We see that any two corresponding infinitesimal figures on the two surfaces can be considered as similar, since the lengths of the arcs are proportional and the angles equal; it is on this account that the term *conformal representation* is often given to every correspondence which does not alter the angles.

Given two surfaces Σ, Σ' and a definite relation which establishes a point-to-point correspondence between these two surfaces, we can always determine whether the conditions (42) are satisfied or not, and therefore whether we have a conformal representation of one of the surfaces on the other. par But we may consider other problems. For example, given the surfaces Σ and Σ', we may propose the problem of determining all the correspondences between the points of the two surfaces which preserve the angles. Suppose that the coördinates (x,y,z) of a point of Σ are expressed as functions of two parameters (u,v), and that the coördinates (x',y',z') of a point of Σ' are expressed as functions of two other parameters (u',v'). Let

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \qquad ds'^2 = E'du'^2 + 2F'du'dv' + G'dv'^2$$

be the expressions for the squares of the linear elements. The problem in question amounts to this: *To find two functions $u' = \pi_1(u,v)$, $v' = \pi_2(u,v)$ such that we have identically*

$$E'd\pi_1^2 + 2F'd\pi_1 d\pi_2 + G'd\pi_2^2 = \lambda^2(E^2du^2 + 2Fdudv + Gdv^2),$$

λ being any function of the variables u, v. The general theory of differential equations shows that this problem always admits an infinite number of solutions; we shall consider only certain special cases.

21. Conformal representation of one plane on another plane. Every correspondence between the points of two planes is defined by relations such as

$$X = P(x, y), \qquad Y = Q(x, y) \tag{44}$$

[[49]] where the two planes are referred to systems of rectangular coördinates (x, y) and (X, Y). From what we have just seen, in order that this transformation shall preserve the angles, it is necessary and sufficient that we have [[49]]

$$dX^2 + dY^2 = \lambda^2(dx^2 + dy^2),$$

where λ is any function whatever of x, y independent of the differentials. Developing the differentials dX, dY and comparing the two sides, we find that the two functions $P(x, y)$ and $Q(x, y)$ must satisfy the two relations

$$\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial Q}{\partial x}\right)^2 = \left(\frac{\partial P}{\partial y}\right)^2 + \left(\frac{\partial Q}{\partial y}\right)^2, \qquad \frac{\partial P}{\partial x}\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\frac{\partial Q}{\partial y} = 0. \tag{45}$$

The partial derivatives $\partial P/\partial y$, $\partial Q/\partial y$ cannot both be zero, for the first of the relations (45) would give also $\partial Q/\partial x = \partial P/\partial x = 0$, and the functions P and Q would be constants. Consequently we can write according to the last relation,

$$\frac{\partial P}{\partial x} = \mu\frac{\partial Q}{\partial y}, \qquad \frac{\partial Q}{\partial x} = -\mu\frac{\partial P}{\partial y},$$

where μ is an auxiliary unknown. Putting these values in the first condition (45), it becomes

$$(\mu^2 - 1)\left[\left(\frac{\partial P}{\partial y}\right)^2 + \left(\frac{\partial Q}{\partial y}\right)^2\right] = 0,$$

and from it we derive the result $\mu = \pm 1$. We must then have either

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \tag{46}$$

or

$$\frac{\partial P}{\partial x} = -\frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{47}$$

The first set of conditions state that $P + Qi$ is an analytic function of $x + yi$. As for the second set, we can reduce it to the first by changing Q to $-Q$, that is, by taking the figure symmetric to the transformed figure with respect to the axis OX. Thus we see, finally, that to every conformal representation of a plane on a plane there corresponds a solution of the system (46), and consequently an analytic function. If we suppose the axes OX and OY parallel respectively to the axes Ox and Oy, the sense of rotation of the angles is preserved or not, according as the functions P and Q satisfy the relations (46) or (47).

[[50]] **22. Riemann's theorem.** Given in the plane of the variable z a region A bounded by a single curve (or simple boundary), and in the plane of the variable u a circle C, Riemann proved that there exists an analytic function $u = f(z)$, analytic in the region A, such that to each point of the region A corresponds a point of the circle, and that, conversely, to a point of the circle corresponds one and only one point of A. The function $f(z)$ depends also upon three arbitrary real constants, which we can dispose of in such a way that the center of the circle corresponds to a given point of the region A, while an arbitrarily chose point on the circumference corresponds to a given point of the boundary of A. We shall not give here the demonstration of this theorem, of which we shall indicate only some examples. [[50]]

We shall point out only that the circle can be replaced by a half-plane. Thus, let us suppose that, in the plane of u, the circumference passes through the origin; the transformation $u' = 1/u$ replaces that circumference by a straight line, and the circle itself by the portion of the u'-plane situated on one side of the straight line extended indefinitely in both directions.

Example 1. Let $u = z^{1/\alpha}$, where α is real and positive. Consider the portion A of the plane included between the direction Ox and a ray through the origin making an angle of $\alpha\pi$ with Ox ($\alpha \leq 2$). Let $z = re^{i\theta}$, $u = Re^{i\omega}$; we have

$$R = r^{\frac{1}{\alpha}}, \qquad \omega = \frac{\theta}{\alpha}.$$

When the point z describes the portion A of the plane, r varies from 0 to $+\infty$ and θ from 0 to $\alpha\pi$; hence R varies from 0 to $+\infty$ and ω from 0 to π. The point u therefore describes the half-plane situated above the axis OX, and to a point of that

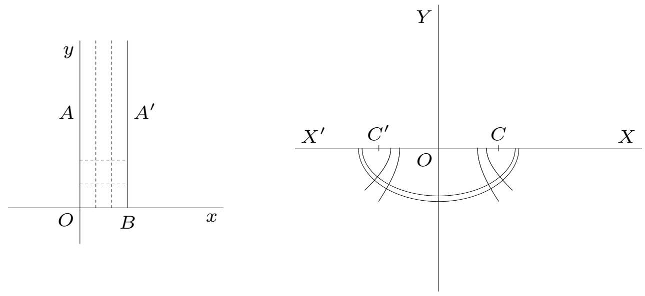

FIG. 11

half-plane corresponds only one point of A, for we have, inversely, $r = R^{\alpha}$, $\theta = \alpha\omega$.

Let us next take the portion B of the z-plane bounded by two arc s of circles which intersect. Let z_0, z_1 be the points of intersection; if we carry out first the transformation

$$z' = \frac{z - z_0}{z - z_1},$$

the region B goes over into a portion A of the z'-plane included between two rays from the origin, for along the arc of a circle [[51]] passing through the points z_0, z_1, the angle of $(z - z_0)/(z - z_1)$ remains constant. Applying now the preceding transformation [[51]] $u = (z')^{1/\alpha}$, we see that the function

$$u = \left(\frac{z - z_0}{z - z_1}\right)^{\frac{1}{\alpha}}$$

enables us to realize the conformal representation of the region B on a half-plane by suitably choosing α.

Example 2. Let $u = \cos z$. Let us cause z to describe the infinite half-strip R, or $AOBA'$ (Fig. 11), defined by the inequalities $0 \leq x \leq \pi$, $y \geq 0$, and let us examine the region described by the point $u = X + Yi$. We have here (§12)

$$X = \cos x \frac{e^y + e^{-y}}{2}, \qquad Y = -\sin x \frac{e^y - e^{-y}}{2}. \tag{48}$$

When x varies from 0 to π, Y is always negative and the point u remains in the half-plane below the axis $X'OX$. Hence, to every point of the region R corresponds a point of the u half-plane, and when the point z is on the boundary of R, we have $Y = 0$, for one of the two factors $\sin x$ or $(e^y - e^{-y})/2$ is zero. Conversely, to every point of the u half-plane below OX corresponds one and only one point of the strip R in the z-plane. In fact, if z' is a root of the equation $u = \cos z$, all the other roots are included in the expression $2k\pi \pm z'$. If the coefficient of i in z' is positive, there cannot be but one of these points in the strip R, for all the points $2k\pi - z'$ are below Ox. There is always one of the points $wk\pi + z'$ situated in R, for there is always one of these points whose abscisse lies between 0 and 2π. That abscissa cannot be included between π and 2π, for the corresponding value of Y would then be positive. The point is therefore located in R.

It is easily seen from the formulæ (48) that when the point z describes the portion of a parallel to Ox in R, the point u describes half of an ellipse. When the point z describes a parallel to Oy, the point u describes a half-branch of a hyperbola. All these conics have as foci the points C, C' of the axis OX, with the abscissas $+1$ and -1.

Example 3. Let

$$u = \frac{e^{\frac{\pi z}{2a}} - 1}{e^{\frac{\pi z}{2a}} + 1} \tag{49}$$

where a is real and positive. In order that $|u|$ shall be less than unity, it is easy to show that it is necessary and sufficient, that $\cos[(\pi y)/(2a)] > 0$. If y varies from $-a$ to $+a$, we see that to the infinite strip included between the two straight lines $y = -a$, $y = +a$ corresponds in the u-plane the circle C described about the origin as center with unit radius. Conversely, to every point of this circle corresponds one and only one point of the infinite strip, for the values of z which correspond to a given value of u form an arithmetical progression with the constant difference of $4ai$. hence there cannot be more than one value of z in the strip considered. Moreover, there is always one of these roots in which the coefficient of i lies between $-a$ and $3a$, and that coefficient cannot lie between a and $3a$, for the corresponding value of $|u|$ would then be greater than *unity*.

[[52]] **23. Geographic maps.** To make a conformal map of a surface means to make the points of the surface correspond to those of a plane in such a way that the angles are unaltered. Suppose that the coördinates of a point of the surface Σ under consideration be expressed as functions of two variable parameters (u, v), and let [[52]]

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

be the square of the linear element for this surface. Let (α, β) be the rectangular coördinates of the point of the plane P which corresponds to the point (u, v) of the surface. The problem here is to find two functions

$$u = \pi_1(\alpha, \beta), \qquad v = \pi_2(\alpha, \beta)$$

of such a nature that we have identically

$$Edu^2 + 2Fdudv + Gdv^2 = \lambda(d\alpha^2 + d\beta^2),$$

where λ is any function whatever of α, β not containing the differentials. This problem admits an infinite number of solutions, which can all be deduced from one of them by means of the conformal transformations, already studied, of one plane on another. Suppose that we actually have at the same time

$$ds^2 = \lambda(d\alpha^2 + d\beta^2), \qquad ds^2 = \lambda'(d\alpha'^2 + d\beta'^2);$$

then we shall also have

$$d\alpha^2 + d\beta^2 = \frac{\lambda'}{\lambda}(d\alpha'^2 + d\beta'^2),$$

so that $\alpha + \beta i$, or $\alpha - \beta i$, will be an analytic function of $\alpha' + \beta' i$. The converse is evident.

Example 1. *Mercator's projection.* We can always make a map of a surface of revolution in such a way that the meridians and the parallels of latitude correspond to the parallels to the axes of coördinates. Thus, let

$$x = \rho\cos\omega, \qquad y = \rho\sin\omega, \qquad z = f(\rho)$$

be the coördinates of a point of a surface of revolution about the axis Oz; we have

$$ds^2 = d\rho^2[1 + f'^2(\rho)] + \rho^2 d\omega^2 = \rho^2\left[d\omega^2 + \frac{1 + f'^2(\rho)}{\rho^2}d\rho^2\right],$$

which can be written

$$ds^2 = \rho^2(dX^2 + dY^2)$$

if we set

$$X = \omega, Y = \int \frac{\sqrt{1 + f'^2(\rho)}}{\rho}\, d\rho.$$

[[53]] In the case of a sphere of radius R we can write the coördinates in the form [[53]]

$$X = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi, \qquad z = R\cos\theta,$$
$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) = R^2\sin^2\theta\left(d\phi^2 + \frac{d\theta^2}{\sin^2\theta}\right),$$

and we shall set

$$X = \phi, Y = \int \frac{d\theta}{\sin\theta} = \log\left(\tan\frac{\theta}{2}\right).$$

We obtain thus what is called *Mercator's projection*, in which the meridians are represented by parallels to the axis OY, and the parallels of latitude by segments of straight lines parallel to OX. To obtain the whole surface of the sphere it is sufficient to let ϕ vary from 0 to 2π, and θ from 0 to π; then X varies from 0 to 2π and Y from $-\infty$ to $+\infty$. The map has then the appearance of an infinite strip of breadth 2π. The curves on the surface of the sphere which cut the meridians at a constant angle are called *loxodromic curves* or *rhumb lines*, and are represented on the map by straight lines.

Example 2. *Stereographic projection.* Again, we may write the square of the linear element of the sphere in the form

$$ds^2 = 4\cos^4\frac{\theta}{2}\left(\frac{R^2 d\theta^2}{4\cos^4\frac{\theta}{2}} + R^2\tan^2\frac{\theta}{2}d\phi^2\right),$$

or

$$ds^2 = 4\cos^4\frac{\theta}{2}(d\rho^2 + \rho^2 d\omega^2),$$

if we set

$$\rho = R\tan\frac{\theta}{2}, \qquad \omega = \phi.$$

But $d\rho^2 + \rho^2 d\omega^2$ represents the square of the linear element of the plane in polar coördinates (ρ, ω); hence it is sufficient, in order to obtain a conformal representation of the sphere, to make a point of the plane with polar coördinates (ρ, ω) correspond to the point (θ, ϕ) of the surface of the sphere. It is seen immediately, on drawing the figure, that ρ and ω are the polar coördinates of the stereographic projection of the point (θ, ϕ) of the sphere on the plane of the equator, the center of projection being one of the poles.*

[[54]] *Example* 3. *Map of an anchor ring.* Consider the anchor ring generated by the revolution of a circle of radius R about an axis situated in its own plane at a distance a fro its center, where $a > R$. Taking the axis of revolution for the axis of z, and the median plane of the anchor ring for the xy-plane, we can write the coördinates of a point of the surface in the form [[54]]

$$z = (a + R\cos\theta)\cos\phi, \qquad y = (a + R\cos\theta)\sin\phi, \qquad z = R\sin\theta,$$

and it is sufficient to let θ and ϕ vary from $-\pi$ to $+\pi$. From these formulæ we deduce

$$ds^2 = (a + R\cos\theta)^2\left[d\phi^2 + \frac{R^2 d\theta^2}{(a + R\cos\theta)^2}\right];$$

and, to obtain a map of the surface, we may set

$$X = \phi,$$
$$Y = e\int_0^\theta \frac{d\theta}{1 + e\cos\theta} = \frac{2e}{\sqrt{1 - e^2}}\text{arc tan}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\theta}{2}\right),$$

where

$$e = \frac{R}{a} < 1.$$

Thus the total surface of the anchor ring corresponds point by point to that of a rectangle whose sides are 2π and $2\pi e/\sqrt{1 - e^2}$.

24. Isothermal curves. Let $U(x, y)$ be a solution of Laplace's equation

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0;$$

the curves represented by the equation

$$U(x, y) = C, \tag{50}$$

where C is an arbitrary constant, form a family of *isothermal* curves. With every solution $U(x, y)$ of Laplace's equation we can associate another solution, $V(x, y)$, such that $U + Vi$ is an analytic function of $x + yi$. The relations

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

show that the two families of isothermal curves

$$U(x, y) = C, \qquad V(x, y) = C'$$

* The center of projection is the south pole if θ is measured from the north pole to the radius. Using the north pole as the center of projection, the point $(R^2/\rho, \omega)$, symmetric to the first point (see Ex. 17, p. 58), would be obtained. – TRANS.

are orthogonal, for the slopes of the tangents to the two curves C and C' are respectively

$$-\frac{\partial U}{\partial x} \div \frac{\partial U}{\partial y}, \qquad -\frac{\partial V}{\partial x} \div \frac{\partial V}{\partial y}.$$

Thus the orthogonal trajectories of a family of isothermal curves form another family of isothermal curves. We obtain all the conjugate systems of isothermal curves by considering all analytic functions $f(z)$ and taking the curves for which the real part of $f(z)$ and the coefficient of i have constant values. The curves for which the absolute value R and the angle Ω of $f(z)$ remain constant also form two conjugate isothermal systems; for the real part of the analytic function $\operatorname{Log}[f(z)]$ is $\log R$, and the coefficient of i is Ω.

Likewise we obtain conjugate isothermal systems by considering the curves described by the point whose coördinates are
[[55]] X, Y, where $f(x) = X + Yi$, when we give to x and y constant values. This is seen by regarding $x + yi$ as an analytic function [[55]]
of $X + Yi$. More generally, every transformation of the points of one plane on the other, which preserves the angles, changes one family of isothermal curves into a new family of isothermal curves. Let

$$x = p(x', y'), \qquad y = q(x', y')$$

be equations defining a transformation which preserves angles, and let $F(x', y')$ be the result obtained on substituting $p(x', y')$ and $q(x', y')$ for x and y in $U(x, y)$. The proof consists in showing that $F(x', y')$ is a solution of Laplace's equation, provided that $U(x, y)$ is a solution. The verification of this fact does not offer any difficulty (see Vol. I, Chap. III, Ex. 8, 2d ed.; Chap. II, Ex. 9, 1st ed.), but the theorem can be established without any calculation. Thus, we can suppose that the functions $p(x', y')$ and $q(x', y')$ satisfy the relations

$$\frac{\partial p}{\partial x'} = \frac{\partial q}{\partial y'}, \qquad \frac{\partial p}{\partial y'} = -\frac{\partial q}{\partial x'},$$

for a symmetric transformation evidently changes a family of isothermal curves into a new family of isothermal curves. The function $x + yi = p + qi$ is then an analytic function of $z' = x' + y'i$, and, after the substitution, $U + Vi$ also becomes an analytic function $F(x', y') + i\Phi(x', y')$ of the same variable z' (§5). Hence the two families of curves

$$F(x', y') + C, \qquad \Phi(x', y') = C'$$

gives a new orthogonal net formed by two conjugate isothermal families.

For example, concentric circles and the rays from the center form two conjugate isothermal families, as we see at once by considering the analytic function $\operatorname{Log} z$. Carrying out an inversion, we have the result that the circles passing through two fixed points also form an isothermal system. The conjugate system is also composed of circles.

Likewise, confocal ellipses form an isothermal system. Indeed, we have seen above that the point $u = \cos z$ describes confocal ellipses when the point z is made to describe parallels to the axis Ox (§22). The conjugate system is made up of confocal and orthogonal hyperbolas.

Note. In order tha a family of curves represented by an equation $P(x, y) = C$ may be isothermal, it is not necessary that the function $P(x, y)$ be a solution of Laplace's equation. Indeed, these curves are represented also by the equation $\phi[P(x, y)] = C$, whatever be the function ϕ; hence it is sufficient to take for the function ϕ a form such that $U(x, y) = \phi(P)$ satisfies Laplace's equation. Making the calculation, we find that we must have

$$\frac{d^2\phi}{dP^2}\left[\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2\right] + \frac{d\phi}{dP}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) = 0;$$

hence it is necessary that the quotient

$$\frac{\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}}{\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2}$$

depend only on P, and if that condition is satisfied, the function ϕ can be obtained by two quadratures.

[[56]] [[56]]

EXERCISES

1. Determine the analytic function $f(x) = X + Yi$ whose real part X is equal to

$$\frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}.$$

Consider the same question, given that $X + Y$ is equal to the preceding function.

2. Let $\phi(m, p) = 0$ be the tangential equation of a real algebraic curve, that is to say, the condition that the straight line $y = mx + p$ be tangent to that curve. The roots of the equation $\phi(i, -zi) = 0$ are the real foci of the curve.

3. If p and q are two integers prime to each other, the two expressions $\left(\sqrt[q]{z}\right)^p$ and $\sqrt[q]{z^p}$ are equivalent. What happens when p and q have a greatest common divisor $d > 1$?

4. Find the absolute value and the angle of e^{x+yi} by considering it as the limit of the polynomial $[1+(x+yi)/m]^m$ when the integer m increases indefinitely.

5. Prove the formula

$$\cos a + \cos(a+b) + \cdots + \cos(a+nb) = \frac{\sin\left(\frac{n+1}{2}b\right)}{\sin\left(\frac{b}{2}\right)} \cos\left(a + \frac{nb}{2}\right),$$

$$\sin a + \sin(a+b) + \cdots + \sin(a+nb) = \frac{\sin\left(\frac{n+1}{2}b\right)}{\sin\left(\frac{b}{2}\right)} \sin\left(a + \frac{nb}{2}\right).$$

6. What is the final value of arc $\sin z$ when the variable z describes the segment of a straight line from the origin to the point $1+i$, if the initial value of arc $\sin z$ is taken as 0?

7. Prove the continuity of a power series by means of the formula (12) (§8)

$$f(z+h) - f(z) = hf_1(z) + \frac{h^2}{2!} f_2(z) + \cdots + \frac{h^n}{n!} f_n(z) + \cdots.$$

[Take a suitable dominant function for the series of the right-hand side.]

8. Calculate the integrals

$$\int x^m e^{eax} \cos bx\, dx, \qquad \int x^m e^{ax} \sin bx\, dx,$$

$$\int \mathrm{ctn}\,(x-a)\mathrm{ctn}\,(x-b)\cdots \mathrm{ctn}\,(x-l)\, dx.$$

9. Given in the plane xOy a closed curve C having any number whatever of double points and described in a determined sense, a numerical coefficient is assigned to each region of the plane determined by the curve according to the rule of Volume I (§97, 2d ed.; §96, 1st ed). Thus, let R, R' be two contiguous regions separated by the arc ab of the curve described in the sense of a to b; the coefficient of the region to the left is greater by unity than the coefficient of the region to the right, and the region exterior to the curve has the coefficient .

[[57]] Let z_0 be a point taken in one of the regions and N the corresponding coefficient. Prove that $2N\pi$ represents the variation of the angle of $z - z_0$ when the point z describes the curve C in the sense chosen. [[57]]

10. By studying the development of Log $[(1+z)/(1-z)]$ on the circle of convergence, prove that the sum of the series

$$\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots + \frac{\sin(2n+1)\theta}{2n+1} + \cdots$$

is equal to $\pm\pi/4$, according as $\sin\theta \gtrless 0$. (Cf. Vol. I, §204, 2d ed.; §198, 1st ed.)

11. Study the curves described by the point $Z = z^2$ when the point z describes a straight line or a circle.

12. The relation $2Z = z + c^2/z$ effects the conformal representation of the region inclosed between two confocal ellipses on the ring-shaped region bounded by two concentric circles.

[Take, for example, $z = Z + \sqrt{Z^2 - c^2}$, make in the Z-plane a straight-line cut $(-c, c)$, and choose for the radical a positive value when Z is real and greater than c.]

13. Every circular transformation $z' = (az+b)/(cz+d)$ can be obtained by the combination of an *even* number of inversions. Prove also the converse.

14. Every transformation defined by the relation $z' = (az_0+b)/(cz_0+d)$, where z_0 indicates the conjugate of z, results from an *odd* number of inversions. Prove also the converse.

15. Fuchsian transformations. Every linear transformation (§19, Ex. 2) $z' = (az+b)/(cz+d)$, where a, b, c, d are real numbers satisfying the relation $ad - bc = 1$, is called a *Fuchsian transformation.* Such a transformation sets up a correspondence such that to every point z situated above Ox corresponds a point z' situated on the same side of Ox'.

The two definite integrals

$$\int \frac{\sqrt{dx^2 + dy^2}}{y}, \qquad \iint \frac{dx\, dy}{y^2}$$

are *invariants* with respect to all these transformations.

The preceding transformation has two double points which correspond to the roots α, β of the equation $cz^2+(d-a)z-b=0$. If α and β are real and distinct, we can write the equation $z' = (az+b)/(cz+d)$ in the equivalent form

$$\frac{z'-\alpha}{z'-\beta} = k\frac{z-\alpha}{z-\beta}$$

where k is real. Such a transformation is called *hyperbolic*.

If α and β are conjugate imaginaries, we can write the equation

$$\frac{z'-\alpha}{z'-\beta} = e^{i\omega}\frac{z-\alpha}{z-\beta}$$

where ω is real. Such a transformation is called *elliptic*.

If $\beta = \alpha$, we can write

$$\frac{1}{z'-\alpha} = \frac{1}{z-\alpha} + k,$$

where α and k are real. Such a transformation is called *parabolic*.

[[58]] **16**. Let $z' = f(z)$ be a Fuchsian transformation. Put [[58]]

$$z_1 = f(z), \qquad z_2 = f(z_1), \qquad \cdots, \qquad z_n = f(z_{n-1}).$$

Prove that all the points z, z_1, z_2, $\cdots$, z_n are on the circumference of a circle. Does the point z_n approach a limiting position as n increases indefinitely?

17. Given a circle C with the center O and radius R, two points M, M' situated on a ray from the center O are said to be *symmetric* with respect to that circle if $OM \times OM' = R^2$.

Let now C, C' be two circles in the same plane and M any point whatever in that plane. Take the point M_1 symmetric to M with respect to the circle C, then the point M_1' symmetric to M_1 with respect to C_1', then the point M_2 symmetric to M_1' with respect to C, and so on forever. Study the distribution of the points M_1, M_1', M_2, M_2', $\cdots$.

18. Find the analytic function $Z = f(z)$ which enables us to pass from Mercator's projection to the stereographic projection.

19*. All the isothermal families composed of circles are made up of circles passing through two fixed points, distinct or coincident, real or imaginary.

[Setting $z = x+yi$, $z_0 = x-yi$, the equation of a family of circles depending upon a single parameter λ may be written in the form

$$zz_0 + az + bz_0 + c = 0,$$

where a, b, c are functions of the parameter λ. In order that this family be isothermal, it is necessary that $\frac{\partial^2\lambda}{\partial z\partial z_0} = 0$. Making the calculation, the theorem stated is proved.]

20*. If $|q| < 1$, we have the identity

$$(1+q)(1+q^2)\cdots(1+q^n)\cdots = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n+1})\cdots},$$ [Euler.

[In order to prove this, transform the infinite product on the left into an infinite product with two indices by putting in the first row the factors $1+q$, $1+q^2$, $1+q^4$, $\cdots$, $1+q^{2^n}$, $\cdots$; in the second row the factors $1+q^3$, $1+q^6$, $\cdots$, $1+(q^3)^{2^n}$, $\cdots$; and then apply the formula (16) of the text.]

21. Develop in powers of z the infinite products

$$F(z) = (1+xz)(1+x^2z)\cdots(1+x^nz)\cdots,$$
$$\Phi(z) = (1+xz)(1+x^8z)\cdots(1+x^{2n+1}z)\cdots.$$

[It is possible, for example, to make use of the relation

$$F(xz)(1+xz) = F(z), \qquad \Phi(x^2z)(1+xz) = \Phi(z).]$$

22*. Supposing $|z| < 1$, prove Euler's formula

$$(1-x)(1-x^2)(1-x^3)\cdots(1-x^n)\cdots = 1-x-x^2+x^5-x^7+x^{12}-\cdots+x^{\frac{3n^2-n}{2}}-x^{\frac{3n^2+n}{2}}+\cdots.$$

(See J. Bertrand, *Calcul différentiel*, p. 328.)

[[59]] **23***. Given a sphere of unit radius, the stereographic projection of that sphere is made on the plane of the equator, the center of projection being one of the poles. To a point M of the sphere is made to correspond the complex number $s = x + yi$, where x and y are the rectangular coördinates of the projection m of M with respect to two rectangular axes of the plane of the equator, the origin being the center of the sphere. To two diametrically opposite points of the sphere correspond two complex numbers, s, $-1/s_0$, where s_0 is the conjugate imaginary to s. Every linear transformation of the form [[59]]

$$\frac{s' - \alpha}{s' - \beta} = e^{i\omega}\frac{s - \alpha}{s - \beta}, \qquad (A)$$

where $\beta\alpha_0 + 1 = 0$, defines a rotation of the sphere about a diameter. To groups of rotations which make a regular polyhedron coincide with itself correspond the groups of finite order of linear substitutions of the form (A). (See Klein, *Das Ikosaeder.*)

CHAPTER II

THE GENERAL THEORY OF ANALYTIC FUNCTIONS ACCORDING TO CAUCHY

I. DEFINITE INTEGRALS TAKEN BETWEEN IMAGINARY LIMITS

25. Definitions and general principles. The results presented in the preceding chapter are independent of the work of Cauchy and, for the most part, prior to that work. We shall now make a systematic study of analytic functions, and determine the logical consequences of the definition of such functions. Let us recall that a function $f(z)$ is analytic in a region A: 1) if to every point taken in the region A corresponds a definite value of $f(z)$; 2) if that value varies continuously with z; 3) if for every point z taken in A the quotient

$$\frac{f(z+h)-f(z)}{h}$$

approaches a limit $f'(z)$ when the absolute value of h approaches zero.

The consideration of definite integrals, when the variable passes through a succession of complex values, is due to Cauchy*; it was the origin of new and fruitful methods.

Let $f(z)$ be a continuous function of z along the curve AMB (Fig. 12). Let us mark off on this curve a certain number of points of division $z_0, z_1, z_2, \cdots, z_{n-1}, z'$, which follow each other in the order of increasing indices when the arc is traversed from a to b, the points z_0 and z' coinciding with the extremities A and B.

Let us take next a second series of points $\zeta_1, \zeta_2, \cdots, \zeta_n$ on the arc AB, the point ζ_k being situated on the arc $z_{k-1}z_k$, and let us consider the sum

$$S = f(\zeta_1)(z_1 - z_0) + f(\zeta_2)(z_2 - z_2) + \cdots + f(\zeta_k)(z_k - z_{k-1}) + \cdots + f(\zeta_n)(z' - z_{n-1}).$$

When the number of points of division $z_1, \cdots, z_{n-1}$ increases indefinitely in such a way that the absolute
[[61]] values of all the differences $z_1 - z_0, z_2 - z_1, \cdots$ become and remain smaller than any positive number arbitrarily [[61]]
chosen, the sum S approaches a limit, which is called the definite integral of $f(z)$ taken along AMB and which is represented by the symbol

$$\int_{(AMB)} f(z)\,dz.$$

To prove this, let us separate the real part and the coefficient of i in S, and let us set

$$f(z) = X + Yi, \qquad z_k = x_k + y_k i, \qquad \zeta_k = \xi_k + \eta_k i,$$

where X and Y are continuous functions along AMB. Uniting the similar terms, we can write the sum S in

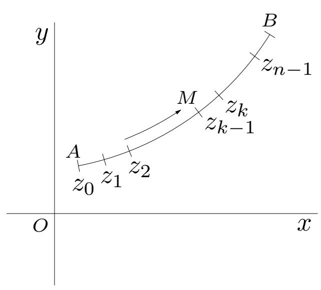

FIG. 12

* *Mémoire sur les intégrals définies, prises entre des limites imaginaires*, 1825. This memoir is reprinted in Volumes VII and VIII of the *Bulletin des Sciences mathématiques* (1st series).

the form

$$\begin{aligned}S =&X(\xi_1,\eta_1)(x_1-x_0)+\cdots+X(\xi_k,\eta_k)(x_k-x_{k-1})+\cdots+X(\xi_n,\eta_n)(x'-x_{n-1})\\&-[Y(\xi_1,\eta_1)(y_1-y_0)+\cdots+Y(\xi_k,\eta_k)(y_k-y_{k-1})+\cdots]\\&+i[X(\xi_1,\eta_1)(y_1-y_0)+\cdots]+i[Y(\xi_1,\eta_1)(x_1-x_0)+\cdots].\end{aligned}$$

When the number of divisions increases indefinitely, the sum of the terms in the same row has for its limit a line integral taken along AMB, and the limit of S is equal to the sum of four line integrals:*

$$\int_{(AMB)} f(z)\,dz = \int_{(AMB)} (X\,dx - Y\,dy) + i\int_{(AMB)} (Y\,dx + X\,dy).$$

[[62]] From the definition it results immediately that [[62]]

$$\int_{(AMB)} f(z)\,dz + \int_{(BMA)} f(z)\,dz = 0.$$

It is often important to know an upper bound for the absolute value of in integral. Let s be the length of the arc AM, L the length of the arc AB, s_{k-1}, s_k, σ the lengths of the arcs $Az_{k-1}, Az_k, A\zeta_k$ of the path of integration. Setting $F(s) = |f(z)|$, we have

$$|f(\zeta_k)(z_k - z_{k-1})| = F(\sigma_k)\,|z_k - z_{k-1}| \le F(\sigma_k)\,(s_k - s_{k-1})\,,$$

for $|z_k - z_{k-1}|$ represents the length of the chord, and $s_k - s_{k-1}$ the length of the arc. Hence the absolute value of S is less than or at most equal to the sum $\sum F(\sigma_k)(s_k - s_{k-1})$; whence, passing to the limit, we find

$$\left|\int_{(AMB)} f(z)\,dz\right| \le \int_0^L F(s)\,ds.$$

Let M be an upper bound for the absolute value of $f(z)$ along the curve AB. It is clear that the absolute value of the integral on the right is less than ML, and we have, a fortiori,

$$\left|\int_{(AMB)} f(z)\,dz\right| < ML.$$

26. Change of variables. Let us consider the case that occurs frequently in applications, in which the coördinates x, y of a point of the arc AB are continuous functions of a variable parameter t, $x = \phi(t), y = \psi(t)$, possessing continuous derivatives $\phi'(t), \psi'(t)$; and let us suppose that the point (x, y) describes the path of integration from A to B as t varies from α to β. Let $P(t)$ and $Q(t)$ be the functions of t obtained by substituting $\phi(t)$ and $\psi(t)$, respectively, for x and y in X and Y.

By the formula established for line integrals (I, §95, 2d ed.; §93, 1st ed.) we have

$$\begin{aligned}\int_{(AB)} X\,dx - Y\,dy &= \int_\alpha^\beta [P(t)\phi'(t) - Q(t)\psi'(t)]\,dt,\\ \int_{(AB)} X\,dy + Y\,dx &= \int_\alpha^\beta [P(t)\psi'(t) + Q(t)\phi'(t)]\,dt.\end{aligned}$$

---

* In order to avoid useless complications in the proofs, we suppose that the coördinates x, y of a point of the arc AMB are continuous functions $x = \phi(t), y = \psi(t)$ of a parameter t, which have only a finite number of maxima and minima between A and B. We can then break up the path of integration into a finite number of arcs which are each represented by an equation of the form $y = F(x)$, the function F being continuous between the corresponding limits; or into a finite number of arcs which are each represented by an equation of the form $x = G(y)$. There is no disadvantage in making this hypothesis, for in all the applications there is always a certain amount of freedom in the choice of the path of integration. Moreover, it would suffice to suppose that $\phi(x)$ and $\psi(x)$ are functions of limited variation. We have seen that in this case the curve AMB is then rectifiable (I, ftns., §§73, 82, 95, 2d ed.).

Adding these two relations, after having multiplied the two sides of the second by i, we obtain

$$\int_{(AB)} f(z)\,dz = \int_{\alpha}^{\beta} [P(t) + iQ(t)][\phi'(t) + i\psi'(t)]\,dt. \tag{1}$$

[[63]] This is precisely the result obtained by applying to the integral $\int f(z)\,dz$ the formula established for definite integrals in the case of real functions of real variables; that is, in order to calculate the integral $\int f(z)\,dz$ we need only substitute $\phi(t) + i\psi(t)$ for z and $[\phi'(t) + i\psi'(t)]\,dt$ for dz in $f(z)\,dz$. The evaluation of $\int f(z)\,dz$ is thus reduced to the evaluation of two ordinary definite integrals. If the path AMB is composed of several pieces of distinct curves, the formula should be applied to each of these pieces separately. [[63]]

Let us consider, for example, the definite integral

$$\int_{-1}^{+1} \frac{dz}{z^2}.$$

We cannot integrate along the axis of reals, since the function to be integrated becomes infinite for $z = 0$, but we can follow any path whatever which does not pass through the origin. Let z describe a semicircle of unit radius about the origin as center. This path is given by setting $z = e^{ti}$ and letting t vary from π to 0. Then the integral takes the form

$$\int_{-1}^{+1} \frac{dz}{z^2} = \int_{\pi}^{0} ie^{-ti}\,dt = i\int_{\pi}^{0} \cos t\,dt + \int_{\pi}^{0} \sin t\,dt = -2.$$

This is precisely the result that would be obtained by substituting the limits of integration directly in the primitive function $-1/z$ according to the fundamental formula of the integral calculus (I, §78, 2d ed.; §76, 1st ed.).

More generally, let $z = \phi(u)$ be a continuous function of a new complex variable $u = \xi + \eta i$ such that, when u describes in its plane a path CND, the variable z describes the curve AMB. To the points of division of the curve AMB correspond on the curve CND the points of division $u_0, u_1, x_2, \cdots, u_{k-1}, u_k, \cdots, u'$. If the function $\phi(u)$ possesses a derivative $\phi'(u)$ along the curve CND, we can write

$$\frac{z_k - z_{k-1}}{u_k - u_{k-1}} = \phi'(u_{k-1}) + \epsilon_k,$$

where ϵ_k approaches zero when u_k approaches u_{k-1} along the curve CND. Taking $\zeta_{k-1} = z_{k-1}$ and replacing $z_k - z_{k-1}$ by the expression derived from the preceding equality, the sum S, considered above, becomes

$$S = \sum_{k=1}^{n} f(z_{k-1})\phi'(u_{k-1})(u_k - u_{k-1}) + \sum_{k=1}^{n} \epsilon_k f(z_{k-1})(u_k - u_{k-1}).$$

The first part of the right-hand side has for its limit the definite integral

$$\int_{(CND)} f[\phi(u)]\phi'(u)\,du.$$

[[64]] As for the remaining term, its absolute value is smaller than $\eta ML'$, where η is a positive number greater than each of the absolute values $|\epsilon_k|$ and where L' is the length of the curve CND. If the points of division can be taken so close that all the absolute values $|\epsilon_k|$ will be less than an arbitrarily chosen positive number, the remaining term will approach zero, and the general formula for the change of variable will be [[64]]

$$\int_{(AMB)} f(z)\,dz = \int_{(CND)} f[\phi(u)]\phi'(u)\,du. \tag{2}$$

This formula is always applicable when $\phi(u)$ is an analytic function; in fact, it will be shown later that the derivative of an analytic function is also an analytic function* (see §34).

* If this property is admitted, the following proposition can easily be proved.

27. The formulæ of Weierstrass and Darboux. The proof of the law of the mean for integrals (I, §76, 2d ed.; §74, 1st ed.) rests upon certain inequalities which cease to have a precise meaning when applied to complex quantities. Weierstrass and Darboux, however, have obtained some interesting results in this connection by considering integrals taken along a segment of the axis of reals. We have seen above that the case of any path whatever can be reduced to this particular case, provided certain mild restrictions are placed upon the path of integration.

Let I be a definite integral of the following form:

$$I = \int_\alpha^\beta f(t)[\phi(t) + i\psi(t)]\, dt,$$

[[65]] where $f(t), \phi(t), \psi(t)$ are three real functions of the real variable t continuous in the interval (α, β). From [[65]] the very definition of the integral we evidently have

$$I = \int_\alpha^\beta f(t)\phi(t)\, dt + i \int_\alpha^\beta f(t)\psi(t)\, dt.$$

Let us suppose, for definiteness, that $\alpha < \beta$; then $t - \alpha$ is the length of the path of integration measured from α, and the general formula which gives an upper bound for the absolute value of a definite integral becomes

$$|I| \leq \int_\alpha^\beta |f(t)[\phi(t) + i\psi(t)]|\, dt,$$

or, supposing that $f(t)$ is positive between α and β,

$$|I| \leq \int_\alpha^\beta f(t)\, |\phi(t) + i\psi(t)|\, dt.$$

Applying the law of the mean to this new integral, and indicating by ξ a value of t lying between α and β, we have also

$$|I| \leq |\phi(\xi) + i\psi(\xi)| \int_\alpha^\beta f(t)\, dt.$$

Setting $F(t) = \phi(t) + i\psi(t)$, this result may also be written in the form

$$I = \lambda F(\xi) \int_\alpha^\beta f(t)\, dt, \tag{3}$$

where λ is a complex number *whose absolute value is less than or equal to unity;* this is Darboux's formula.

Let $f(z)$ be an analytic function in a finite region A of the plane. For every positive number ϵ another positive number η can be found such that

$$\left|\frac{f(z+h) - f(z)}{h} - f'(z)\right| < \epsilon,$$

when z and $z + h$ are two points of A whose distance from each other $|h|$ is less than η.

For, let $f(z) = P(x, y) + iQ(x, y), h = \Delta x + i\Delta y$. From the calculation made in §3, to find the conditions for the existence of a unique derivative, we can write

$$\frac{f(z+h) - f(z)}{h} - f'(z) = \frac{[P'_x(x + \theta\Delta x, y) - P'_x]\Delta x}{\Delta x + i\Delta y} + \frac{[P'_y(x + \Delta x, y + \theta\Delta y) - P'_y(x, y)]\Delta y}{\Delta x + i\Delta y} + \cdots$$

Since the derivatives P'_x, P'_y, Q'_x, Q'_y are continuous in the region A, we can find a number η such that the absolute values of the coefficients of Δx and of Δy are less then $\epsilon/4$, when $\sqrt{\Delta x^2 + \Delta y^2}$ is less than η. Hence the inequality written down above will be satisfied if we have $|h| < \eta$. This being the case, if the function $\phi(u)$ is analytic in the region A, all the absolute values $|\epsilon_k|$ will be smaller than a given positive number ϵ, provided the distance between two consecutive points of division of the curve CND is less than the corresponding number η, and the formula (2) will be established.

To Weierstrass is due a more precise expression, which has a relation to some elementary facts of statics. When t varies from α to β, the point with the coördinates $x = \phi(t)$, $y = \psi(t)$ describes a certain curve L. Let $(x_0, y_0), (x_1, y_1), \cdots, (x_{k-1}, y_{k-1}), \cdots$ be the points of L which correspond to the values $\alpha, t_1, \cdots, t_{k-1}, \cdots$ of t; and let us set

$$X = \frac{\sum \phi(t_{k-1}) f(t_{k-1})(t_k - t_{k-1})}{\sum f(t_{k-1}(t_k - t_{k-1})},$$
$$Y = \frac{\sum \psi(t_{k-1}) f(t_{k-1})(t_k - t_{k-1})}{\sum f(t_{k-1})(t_k - t_{k-1})}.$$

According to a known theorem, X and Y are the coördinates of the center of gravity of a system of masses placed at the points $(x_0, y_0), (x_1, y_1), \cdots, (x_{k-1}, y_{k-1}), \cdots$ of the curve L, the mass placed at the point [[66]] (x_{k-1}, y_{k-1}) being equal to $f(t_{k-1})(t_k - t_{k-1})$, where $f(t)$ is still supposed to be positive. It is clear that the center of gravity lies within every closed convex curve C that envelops the curve L. When the number of intervals increases indefinitely, the point (X, Y) will have for its limit a point whose coördinates (u, v) are given by the equations

$$u = \frac{\int_\alpha^\beta f(t)\phi(t)\,dt}{\int_\alpha^\beta f(t)\,dt}, \qquad v = \frac{\int_\alpha^\beta f(t)\psi(t)\,dt}{\int_\alpha^\beta f(t)\,dt},$$

which is itself within the curve C. We can state these two formulæ as one by writing

$$(4) \qquad I = (u + iv)\int_\alpha^\beta f(t)\,dt = Z\int_\alpha^\beta f(t)\,dt,$$

where Z is a point of the complex plane *situated within every closed convex curve enveloping the curve L.* It is clear that, in the general case, the factor Z of Weierstrass is limited to a much more restricted region than the factor $\lambda F(\xi)$ of Darboux.

28. Integrals taken along a closed curve. In the preceding paragraphs, it suffices to suppose that $f(z)$ is a continuous function of the complex variable z along the path of integration. We shall now suppose also that $f(z)$ is an analytic function, and we shall first consider how the value of the definite integral is affected by the path followed by the variable in going from A to B.

If a function $f(z)$ is analytic within a closed curve and also on the curve itself, the integral $\int f(z)\,dz$, taken around that curve, is equal to zero.

In order to demonstrate this fundamental theorem, which is due to Cauchy, we shall first establish several lemmas:

1) The integrals $\int dz$, $\int z\,dz$, taken along any closed curve whatever, are zero. In fact, by definition, the integral $\int dz$, taken along any path whatever between the two points a, b, is equal to $b - a$, and the integral is zero if the path is closed, since then $b = a$. As for the integral $\int z\,dz$, taken along any curve whatever joining two points a, b, if we take successively $\zeta_k = z_{k-1}$, then $\zeta_k = z_k$ (§25), we see that the integral is also the limit of the sum

$$\sum_i \frac{z_i(z_{i+1} - z_i) + z_{i+1}(z_{i+1} - z_i)}{2} = \sum_i \frac{z_{i+1}^2 - z_i^2}{2} = \frac{b^2 - a^2}{2};$$

hence it is equal to zero if the curve is closed.

2) If the region bounded by any curve C whatever be divided into smaller parts by transversal curves [[67]] drawn arbitrarily, the sum of the integrals $\int f(z)\,dz$ taken in the same sense along the boundary of each of these parts is equal to the integral $\int f(z)\,dz$ taken along the complete boundary C. It is clear that each portion of the auxiliary curves separates two contiguous regions and must be described twice in integration in opposite senses. Adding all these integrals, there will remain then only the integrals taken along the boundary curve, whose sum is the integral $\int_{(C)} f(z)\,dz$.

Let us now suppose that the region A is divided up, partly in smaller regular parts, which shall be squares having their sides parallel to the axes Ox, Oy; partly in irregular parts, which shall be portions of squares of which the remaining part lies beyond the boundary C. These squares need not necessarily be equal. For example, we might suppose that two sets of parallels to Ox and Oy have been drawn, the distance between two neighboring parallels being constant and equal to l; then some of the squares thus obtained might be divided up into smaller squares by new parallels to the axes. Whatever may be the manner of subdivision adopted, let us suppose that there are N regular parts and N' irregular parts; let us number the regular parts in any order whatever from 1 to N, and the irregular parts from 1 to N'. Let l_i be the length of the side of the ith square and l'_k that of the square to which the kth irregular part belongs, L the length of the boundary C, and $\mathcal{A}$ the area of a polygon which contains within it the curve C.

Let $abcd$ be the ith square (Fig. 13), let z_i be a point taken in its interior or on one of its sides, and let

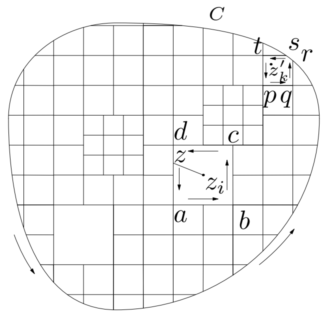

FIG. 13

z be any point on its boundary. Then we have

$$\frac{f(z)-f(z_i)}{z-z_i} = f'(z_i)+\epsilon_i, \tag{5}$$

where $|\epsilon_i|$ is small, provided that the side of the square is itself small. It follows that

$$f(z) = zf'(z_i) + f(z_i) - z_i f'(z_i) + \epsilon_i(z-z_i),$$

$$\int_{(c_i)} f(z)\,dz = f'(z_i)\int_{(c_i)} z\,dz + [f(z_i) - z_i f'(z_i)]\int_{(c_i)} dz + \int_{(c_i)} \epsilon_i(z-z_i)\,dz,$$

[[68]] where the integrals are to be taken along the perimeter c_i of the square. By the first lemma stated above, this reduces to the form

$$\int_{(c_i)} f(z)\,dz = \int_{(c_i)} \epsilon_i(z-z_i)\,dz. \tag{6}$$

Again, let $pqrst$ be the kth irregular part, let z'_k be a point taken in its interior or on its perimeter, and let z be any point of its perimeter. Then we have, as above,

$$\frac{f(z)-f(z'_k)}{z-z'_k} = f'(z'_k)+\epsilon'_k, \tag{7}$$

where ϵ'_k is infinitesimal at the same time as l'_k; whence we find

$$\int_{(c'_k)} f(z)\,dz = \int_{(c'_k)} \epsilon'_k(z-z'_k)\,dz. \tag{8}$$

Let η be a positive number greater than the absolute values of all the factors ϵ_i and ϵ'_k. The absolute value of $z-z_i$ is less than $l_i\sqrt{2}$; hence, by (6), we find

$$\left|\int_{(c_i)} f(z)\,dz\right| < 4l_i^2\eta\sqrt{2} = 4\eta\sqrt{2}\omega_i,$$

where ω_i denotes the area of the ith regular part. From (8) we find, in the same way,

$$\left|\int_{(c'_k)} f(z)\,dz\right| < \eta l'_k\sqrt{2}(4l_k + \operatorname{arc} rs) = 4\eta\sqrt{2}\omega'_k + \eta l'_k\sqrt{2}\operatorname{arc} rs,$$

where ω'_k is the area of the square which contains the kth irregular part. Adding all these integrals, we obtain, a fortiori, the inequality

$$(9) \qquad \left|\int_{(C)} f(z)\,dz\right| < \eta\left[r\sqrt{2}(\sum\omega_i + \sum\omega'_k) + \lambda\sqrt{2}L\right],$$

where λ is an upper bound for the sides l'_k. When the number of squares is increased indefinitely in such a way that all the sides l_i and l'_k approach zero, the sum $\sum\omega_i + \sum\omega'_k$ finally becomes less than $\mathcal{A}$. On the right-hand side of the inequality (9) we have, then, the product of a factor which remains finite and another factor η which can be supposed smaller than any given positive number. This can be true only if the left-hand side is zero; we have then

$$\int_{(C)} f(z)\,dz = 0.$$

[[69]] **29.** In order that the preceding conclusion may be legitimate, we must make sure that we can take the squares so small that the absolute values of all the quantities ϵ_i, ϵ'_k will be less than a positive number η given in advance, if the points z_i and z'_k are suitably chosen.* We shall say for brevity that a region bounded by a closed curve γ, situated in a region of the plane inclosed by the curve C, satisfies the condition (α) with respect to the number η if it is possible to find in the interior of the curve γ or on the curve itself a point z' such that we always have [[69]]

$$(\alpha) \qquad \left|f(z) - f(z') - (z - z')f'(z')\right| \leq |z - z'|\eta,$$

when z describes the curve γ. The proof depends on showing that *we can choose the squares so small that all the parts considered, regular and irregular, satisfy the condition (α) with respect the number η.*

We shall establish this new lemma by the well-known process of successive subdivisions. Suppose that we have first drawn two sets of parallels to the axes Ox, Oy, the distance between two adjacent parallels being constant and equal to l. Of the parts obtained, some may satisfy the condition (α), while others do not. Without changing the parts which do satisfy the condition (α), we shall divide the others into smaller parts by joining the middle points of the opposite sides of the squares which form these parts or which inclose them. If after this new operation, there are still parts which do not satisfy the condition (α), we will repeat the operation on those parts, and so on. Continuing in this way, there can be only two cases: either we shall end by having only regions which satisfy the condition (α), in which case the lemma is proved; or, however far we go in the succession of operations, we shall always find some parts which do not satisfy that condition.

In the latter case, in at least one of the regular or irregular parts obtained by the first division, the process of subdivision just described never leads us to a set of regions all of which satisfy the condition (α); let A_1 be such a part. After the second subdivision, the part A_1 contains at least one subdivision A_2 which cannot be subdivided into regions all of which satisfy the condition (α). Since it is possible to continue this reasoning indefinitely, we shall have a succession of regions

$$A_1, A_2, A_3, \cdots, A_n, \cdots$$

which are squares, or portions of squares, such that each is included in the preceding, and whose dimensions approach zero as n becomes infinite. There is, therefore, a limit point z_0 situated in the interior of the curve or on the curve itself. Since, by hypothesis, the function $f(z)$ possesses a derivative $f'(z_0)$ for $z = z_0$, we can find a number ρ such that

$$\left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| \leq \eta\,|z - z_0|,$$

provided that $|z - z_0|$ is less than ρ. Let c be the circle with radius ρ described about the point z_0 as center. For large enough values of n, the region A_n will lie within the circle c, and we shall have for all the points of the boundary of A_n

$$\left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| \leq |z - z_0|\,\eta.$$

[[70]] Moreover, it is clear that the point z_0 is in the interior of A_n or on the boundary; hence that region must satisfy the condition (α) with respect to η. We are therefore led to a contradiction in supposing that the lemma is not true. [[70]]

30. By means of a suitable convention as to the sense of integration the theorem can be extended also to boundaries formed by several distinct closed curves. Let us consider, for example, a function $f(z)$ analytic within the region A bounded by the closed curve C and the two interior curves C', C'', and on

* Goursat, *Transactions of the American Mathematical Society*, 1900, Vol. I, p. 14.

these curves themselves (Fig. 14).The complete boundary Γ of the region A is formed by these three distinct

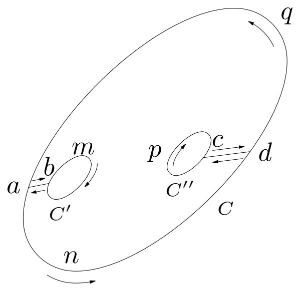

FIG. 14

curves, and we shall say that that boundary is described in the positive sense if the region A is on the left hand with respect to this sense of motion; the arrows on the figure indicate the positive sense of description for each of the curves. With this agreement, we have always

$$\int_{(\Gamma)} f(z)\,dz = 0,$$

the integral being taken along the complete boundary in the positive sense. The proof given for a region with a simple boundary can be applied again here; we can also reduce this case to the preceding by drawing the transversals ab, cd and by applying the theorem to the closed curve *abmbandcpcdqa* (I, §153).

It is sometimes convenient in the applications to write the preceding formula in the form

$$\int_{(C)} f(z)\,dz = \int_{(C')} f(z)\,dz + \int_{(C'')} f(z)\,dz,$$

where the three integrals are now taken in the same sense; that is, the last two must be taken in the reverse direction to that indicated by the arrows.

Let us return to the question proposed at the beginning of §28; the answer is now very easy. Let $f(z)$ be an analytic function in a region $\mathcal{A}$ of the plane. Given two paths AMB, ANB, having the same extremities and lying entirely in that region, they will give the same value for the integral $\int f(z)\,dz$ if the function $f(z)$ is analytic within the closed curve formed by the path AMB followed by the path BNA. We shall suppose,
[[71]] for definiteness, that that closed curve does not have any double points. Indeed, since the sum of the two [[71]]
integrals along AMB and along BNA is zero, the two integrals along AMB and along ANB must be equal. We can state this result again as follows: *Two paths AMB and ANB, having the same extremities, give the same value for the integral $\int f(z)\,dz$ if we can pass from one to the other by a continuous deformation without encountering any point where the function ceases to be analytic.*

This statement holds true even when the two paths have any number whatever of common points besides the two extremities (I, §152). From this we conclude that, when $f(z)$ is analytic in a region bounded by *a single* closed curve, the integral $\int f(z)\,dz$ is equal to zero when taken along any closed curve whatever situated in that region. But we must not apply this result to the case of a region bounded by several distinct closed curves. Let us consider, for example, a function $f(z)$ analytic in the ring-shaped region between two concentric circles C, C'. Let C'' be a circle having the same center and lying between C and C'; the integral $\int f(z)\,dz$, taken along C'', is not in general zero. Cauchy's theorem shows only that the value of that integral remains the same when the radius of the circle C'' is varied.*

* Cauchy's theorem remains true without any hypothesis upon the existence of the function $f(z)$ beyond the region A limited by the curve C, or upon the existence of a derivative at each point of the curve C itself. It is sufficient that the function $f(z)$ shall be analytic at every point of the region A, and *continuous on the boundary C*, that is, that the value $f(Z)$ of the function in a point Z of C varies continuously with the position of Z on that boundary, and that the difference $f(Z) - f(z)$, where z is an interior point, approaches zero uniformly with $|Z - z|$. In fact, let us first suppose that every straight line from a fixed point a of A meets the boundary in a single point. When the point z describes C, the point $a + \theta(z - a)$ (where θ is a real number between 0 and 1) describes a closed curve C' situated in A. The difference between the two

[[72]] **31. Generalization of the formulæ of the integral calculus.** Let $f(z)$ be an analytic function [[72]] in the region A limited by a simple boundary curve C. The definite integral

$$\Phi(Z) = \int_{z_0}^{Z} f(z)\,dz,$$

taken from a fixed point z_0 up to a variable point Z along a path lying in the region A, is, from what we have just seen, a definite function of the upper limit Z. We shall now show that this function $\Phi(z)$ is also an analytic function of Z whose derivative is $f(Z)$. For let $Z+h$ be a point near Z; then we have

$$\Phi(Z+h) - \Phi(Z) = \int_{Z}^{Z+h} f(z)\,dz,$$

and we may suppose that this last integral is taken along the segment of a straight line joining the two points Z and $Z+h$. If the two points are very close together, $f(z)$ differs very little from $f(Z)$ along that path, and we can write

$$f(z) = f(Z) + \delta,$$

where $|\delta|$ is less than any given positive number η, provided that $|h|$ is small enough. Hence we have, after dividing by h,

$$\frac{\Phi(Z+h) - \Phi(Z)}{h} = f(Z) + \frac{1}{h}\int_{Z}^{Z+h} \delta\,dz.$$

The absolute value of the last integral is less than $\eta|h|$, and therefore the left-hand side has for its limit $f(Z)$ when h approaches zero.

If a function $F(Z)$ whose derivative is $f(Z)$ is already known, the two functions $\Phi(z)$ and $F(Z)$ differ only by a constant (footnote, p. 38), and we see that the fundamental formula of integral calculus can be extended to the case of complex variables:

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0). \tag{10}$$

This formula, established by supposing that the two functions $f(z), F(z)$ were analytic in the region A, is applicable in more general cases. It may happen that the function $F(z)$, or both $f(z)$ and $F(z)$ at the same time, are multiple-valued; the integral has a precise meaning if the path of integration does not pass through any of the critical points of these functions. In the application of the formula it will be necessary to pick out an initial determination $F(z_0)$ of the primitive function, and to follow the continuous variation of that
[[73]] function when the variable z describes the path of integration. Moreover, if $f(z)$ is itself a multiple-valued [[73]]

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integrals, along the curves C and C', is equal to

$$\delta = \int_{(C)} \{f(z) - \theta f[z - (z-a)(1-\theta)]\}\,dz,$$

and we can take the difference $1-\theta$ so small that $|\delta|$ will be less than any given positive number, for we can write the function under the integral sign in the form

$$f(z) - f[z - (z-a)(1-\theta)] + (1-\theta)f[z - (z-a)(1-\theta)].$$

Since the integral along C' is zero, we have, then, also

$$\int_{(C)} f(z)\,dz = 0.$$

In the case of a boundary of any form whatever, we can replace this boundary by a succession of closed curves that fulfill the preceding condition by drawing suitably placed transversals.

function, it will be necessary to choose, among the determinations of $F(z)$, that one whose derivative is equal to the determination chosen for $f(z)$.

Whenever the path of integration can be inclosed within a region with a simple boundary, in which the branches of the two functions $f(z), F(z)$ under consideration are analytic, the formula may be regarded as demonstrated. Now in any case, whatever may be the path of integration, we can break it up into several pieces for which the preceding condition is satisfied, and apply the formula (10) to each of them separately. Adding the results, we see that the formula is true in general, provided that we apply it with the necessary precautions.

Let us, for example, calculate the definite integral $\int_{z_0}^{z_1} z^m\,dz$, taken along any path whatever not passing through the origin, where m is a real or a complex number different from -1. One primitive function is $z^{m+1}/(m+1)$, and the general formula (10) gives

$$\int_{z_0}^{z_1} z^m\,dz = \frac{z_1^{m+1} - z_0^{m+1}}{m+1}.$$

In order to remove the ambiguity present in this formula when m is not an integer, let us write it in the form:

$$\int_{z_0}^{z_1} z^m\,dz = \frac{e^{(m+1)\operatorname{Log}(z_1)} - e^{(m+1)\operatorname{Log}(z_0)}}{m+1}.$$

The initial value $\operatorname{Log}(z_0)$ having been chosen, the value of z^m is thereby fixed along the whole path of integration, as is also the final value $\operatorname{Log}(z_1)$. The value of the integral depends both upon the initial value chosen for $\operatorname{Log}(z_0)$ and upon the path of integration. Similarly, the formula

$$\int_{z_0}^{z_1} \frac{f'(z)}{f(z)}\,dz = \operatorname{Log}[f(z_1)] - \operatorname{Log}[f(z_0)]$$

does not present any difficulty in interpretation if the function $f(z)$ is continuous and does not vanish along the path of integration. The point $u = f(z)$ describes in its plane an arc of a curve not passing through the origin, and the right-hand side is equal to the variation of $\operatorname{Log}(u)$ along this arc. Finally, we may remark in passing that the formula for integration by parts, since it is a consequence of the formula (10), can be extended to integrals of functions of a complex variable.

[[74]] **32. Another proof of the preceding results.** The properties of the integral $\int f(z)\,dz$ present a great analogy to the properties of line integrals when the condition for integrability is fulfiled (I, §152). Riemann has shown, in fact, that Cauchy's theorem results immediately from the analogous theorem relative to line integrals. Let $f(z) = X + Yi$ be an analytic function of z within a region A with a simple boundary; the integral taken along a closed curve C lying in that region is the sum of two line integrals: [[74]]

$$\int_{(C)} f(z)\,dz = \int_{(C)} X\,dx - Y\,dy + i\int_{(C)} Y\,dx + X\,dy,$$

and, from the relations which connect the derivatives of the functions X, Y,

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \qquad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x},$$

we see that both of these line integrals are zero* (I, §152).

It follows that the integral $\int_{z_0}^{z} f(z)\,dz$, taken from a fixed point z_0 to a variable point z, is a single-valued function $\Phi(z)$ in the region A. Let us separate the real part and the coefficient of i in that function:

$$\Phi(z) = P(x,y) + iQ(x,y),$$
$$P(x,y) = \int_{(x_0,y_0)}^{(x,y)} X\,dx - Y\,dy, \qquad Q(x,y) = \int_{(x_0,y_0)}^{(x,y)} Y\,dx + X\,dy.$$

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* It should be noted that Riemann's proof assumes the continuity of the derivatives $\partial X/\partial x, \partial Y/\partial y, \cdots$; that is, of $f'(z)$.

The functions P and Q have partial derivatives,

$$\frac{\partial P}{\partial x} = X, \qquad \frac{\partial P}{\partial y} = -Y, \qquad \frac{\partial}{Q}\partial x = Y, \qquad \frac{\partial Q}{\partial y} = X,$$

which satisfy the conditions

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

Consequently, $P + Qi$ is an analytic function of z whose derivative is $X + Yi$ or $f(z)$.

If the function $f(z)$ is discontinuous at a certain number of points of A, the same thing will be true of one or more of the functions X, Y, and the line integrals $P(x,y), Q(x,y)$ will in general have periods that arise from loops described about points of discontinuity (I, §153). The same thing will then be true of the integral $\int_{z_0}^{z} f(z)\,dz$. We shall resume the study of these periods, after having investigated the nature of the singular points of $f(z)$.

[[75]] To give at least one example of this, let us consider the integral $\int_1^z dz/z$. After separating the real part and the coefficient of i, we have [[75]]

$$\int_1^z \frac{dz}{z} = \int_{(1,0)}^{(x,y)} \frac{dx + i dy}{x + iy} = \int_{(1,0)}^{(x,y)} \frac{x\,dx + y\,dy}{x^2 + y^2} + i\int_{(1,0)}^{(x,y)} \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

The real part is equal to $[\log(x^2 + y^2)]/2$, whatever may be the path followed. As for the coefficient of i, we have seen that it has the period 2π; it is equal to the angle through which the radius vector joining the origin to the point (x, y) has turned. We thus find again the various determinations of $\mathrm{Log}\,(z)$.

II. CAUCHY'S INTEGRAL. TAYLOR'S AND LAURENT'S SERIES. SINGULAR POINTS. RESIDUES

We shall now present a series of new and important results, which Cauchy deduced from the consideration of definite integrals taken between imaginary limits.

33. The fundamental formula. Let $f(z)$ be an analytic function in the finite region A limited by a boundary Γ, composed of one or of several distinct closed curves, and continuous on the boundary itself. If x is a point* of the region A, the function

$$\frac{f(z)}{z - x}$$

is analytic in the same region, except at the point $z = x$.

With the point x as center, let us describe a circle γ with the radius ρ, lying entirely in the region a; the preceding function is then analytic in the region of the plane limited by the boundary Γ and the circle γ, and we can apply to it the general theorem (§28). Suppose, for definiteness, that the boundary Γ is composed of two closed curves C, C' (Fig. 15). Then we have

$$\int_{(C)} \frac{f(z)\,dz}{z - x} = \int_{(C')} \frac{f(z)\,dz}{z - x} + \int_{(\gamma)} \frac{f(z)\,dz}{z - x},$$

where the three integrals are taken in the sense indicated by the arrows. We can write this in the form

$$\int_{(\Gamma)} \frac{f(z)\,dz}{z - x} = \int_{(\gamma)} \frac{f(z)\,dz}{z - x},$$

[[76]] where the integral $\int_{(\Gamma)}$ denotes the integral taken along the total boundary Γ in the positive sense. If the radius ρ of the circle γ is very small, the value of $f(z)$ at any point of this circle differs very little from $f(x)$: [[76]]

$$f(z) = f(x) + \delta,$$

* In what follows we shall often have to consider several complex quantities at the same time. We shall denote them indifferently by the letters $x, z, u, \cdots$. Unless it is expressly stated, the letter x will no longer be reserved to denote a real variable.

where $|\delta|$ is very small. Replacing $f(z)$ by this value, we find

$$\int_{(\Gamma)} \frac{f(z)\,dz}{z-x} = f(x)\int_{(\gamma)} \frac{dz}{z-x} + \int_{(\gamma)} \frac{\delta\,dz}{z-x}. \tag{11}$$

The first integral of the right-hand side is easily evaluated; if we put $z = x + \rho e^{\theta i}$, it becomes

$$\int_{(\gamma)} \frac{dz}{z-x} = \int_0^{2\pi} \frac{i\rho e^{\theta i}\,d\theta}{\rho e^{\theta i}} = 2\pi i.$$

The second integral $\int_{(\gamma)} \delta dz/(z-x)$ is therefore independent of the radius ρ of the circle γ; on the other hand, if δ remains less than a positive number η, the absolute value of this integral is less than $(\eta/\rho)2\pi\rho = 2\pi\eta$.

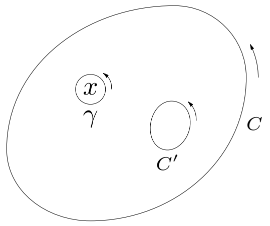

FIG. 15

Now, since the function $f(z)$ is continuous for $z = x$, we can choose the radius ρ so small that η also will be as small as we wish. Hence this integral must be zero. Dividing the two sides of the equation (11) by $2\pi i$, we obtain

$$f(x) = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{z-x}. \tag{12}$$

This is Cauchy's fundamental formula. It expresses the value of the function $f(z)$ at any point x whatever within the boundary by means of the values of the same function taken only along that boundary.

Let $x + \Delta x$ be a point near x, which for example, we shall suppose lies in the interior of the circle γ of radius ρ. Then we have also

$$f(x+\Delta x) = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{z-x-\Delta x},$$

[[77]] and consequently, subtracting the sides of (12) from the corresponding sides of this equation and dividing [[77]]
by Δx, we find

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{(z-x)(z-x-\Delta x)}.$$

When Δx approaches zero, the function under the integral sign approaches the limit $f(z)/(z-x)^2$. In order to prove rigorously that we have the right to apply the usual formula for differentiation, let us write the integral in the form

$$\int_{(\Gamma)} \frac{f(z)\,dz}{(z-x)(z-x-\Delta x)} = \int_{(\Gamma)} \frac{f(z)\,dz}{(z-x)^2} + \int_{(\Gamma)} \frac{\Delta x\, f(z)\,dz}{(z-x)^2(z-x-\Delta x)}.$$

Let M be an upper bound for $|f(z)|$ along Γ, L the length of the boundary, and δ a lower bound for the distance of any point whatever of the circle γ to any point whatever of Γ. The absolute value of the last integral is less than $ML\,|\Delta x|\,/\delta^2$ and consequently approaches zero with $|\Delta x|$. Passing to the limit, we obtain the result

$$f'(x) = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{(z-x)^2}. \tag{13}$$

It may be shown in the same way that the usual method of differentiation under the integral sign can be applied to this new integral* and to all those which can be deduced from it, and we obtain successively

$$f''(x) = \frac{2!}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{(z_x)^3}, \qquad f'''(x) = \frac{3!}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{(z-x)^4},$$

and, in general,

$$f^{(n)}(x) = \frac{n!}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{(z-x)^{n+1}}. \tag{14}$$

Hence, if a function $f(z)$ is analytic in a certain region of the plane, the sequence of successive derivatives of that function is unlimited, and all these derivatives are also analytic functions in the same region. It is to be noticed that we have arrived at this result by assuming only the existence of the first derivative.

Note. The reasoning of this paragraph leads to more general conclusions. Let $\phi(z)$ be a continuous
[[78]] function (but not necessarily analytic) of the complex variable z along the curve Γ, closed or not. The [[78]]
integral

$$F(x) = \int_{(\Gamma)} \frac{\phi(z)\,dz}{z-x}$$

has a definite value for every value of x that does not lie on the path of integration. The evaluations just made prove that the limit of the quotient $[F(x+\Delta x) - F(x)]/\Delta x$ is the definite integral

$$F'(x) = \int_{(\Gamma)} \frac{\phi(z)\,dz}{(z-x)^2},$$

when $|\Delta x|$ approaches zero. Hence $F(x)$ is an analytic function for every value of x, except for the points of the curve Γ, which are in general singular points for that function (see §90). Similarly, we find that the nth derivative $F^{(n)}(x)$ has for its value

$$F^{(n)}(x) = n!\int_{(\Gamma)} \frac{\phi(z)\,dz}{(z-x)^{n+1}}.$$

34. Morera's theorem. A converse of Cauchy's fundamental theorem which was first proved by Morera may be stated as follows: *If a function f(z) of a complex variable z is continuous in a region A, and if the definite integral* $\int_{(C)} f(z)\,dz$*, taken along any closed curve C lying in A, is zero, then f(z) is an analytic function in A.*

For the definite integral $F(z) = \int_{z_0}^{z} f(t)\,dt$, taken between the two points z_0, z of the region A along any path whatever lying in that region, has a definite value independent of the path. If the point z_0 is supposed fixed, the integral is a function of z. The reasoning of §31 shows that the quotient $\Delta F/\Delta z$ has $f(z)$ for its limit when Δz approaches zero. Hence the function $F(z)$ is an analytic function of z having $f(z)$ for its derivative, and that derivative is therefore also an analytic function.

35. Taylor's series. *Let f(z) be an analytic function in the interior of a circle with the center a; the value of that function at any point x within the circle is equal to the sum of the convergent series*

$$f(z) = f(a) + \frac{x-1}{1}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots. \tag{15}$$

In the demonstration we can suppose that the function $f(z)$ is analytic on the circumference of the circle itself; in fact, if x is any point in the interior of the circle C, we can always find a circle C', with center a
[[79]] and with a radius less than that of C, which contains the point x within it, and we would reason with the [[79]]
circle C' just as we are about to do with the circle C. With this understanding, x being an interior point of C, we have, by the fundamental formula,

$$f(x) = \frac{1}{2\pi i}\int_{(C)} \frac{f(z)}{z-x}\,dz. \tag{12'}$$

* The general formula for differentiation under the integral sign will be established later (Chapter V).

Let us now write $1/(z-x)$ in the following way:

$$\frac{1}{z-x} = \frac{1}{z-a-(x-a)} = \frac{1}{z-a}\left(\frac{1}{1-\dfrac{x-a}{z-a}}\right),$$

or, carrying out the division up to the remainder of degree $n+1$ in $x-a$,

$$\frac{1}{z-x} = \frac{1}{z-x} = \frac{1}{z-a} + \frac{x-a}{z-a)^2} + \frac{(x-a)^2}{(z-a)^3} + \cdots + \frac{(x-a)^n}{(x-a)^{n+1}} + \frac{(x-a)^{n+1}}{(z-x)(z-a)^{n+1}}.$$

Let us replace $1/(z-x)$ in the formula (12') by this expression, and let us bring the factors $x-a, (x-a)^2, \cdots$, independent of z, outside of the integral sign. This gives

$$f(x) = J_0 + J_1(x-a) + \cdots + J_n(x-a)^n + R_n,$$

where the coefficients $J_0, J_1, \cdots, J_n$ and the remainder R_n have the values

$$(16) \qquad \begin{cases} J_0 = \dfrac{1}{2\pi i}\displaystyle\int_{(C)} \frac{f(z)\,dz}{z-a}, & J_1 = \dfrac{1}{2\pi i}\displaystyle\int_{(C)} \frac{f(z)\,dz}{(z-a)^2}, \qquad \cdots, \\ J_n = \dfrac{1}{2\pi i}\displaystyle\int_{(C)} \frac{f(z)\,dz}{(z-a)^{n+1}}, & R_n = \dfrac{1}{2\pi i}\displaystyle\int_{(C)} \left(\frac{x-a}{z-a}\right)^{n+1} \frac{f(z)\,dz}{z-x}. \end{cases}$$

As n becomes infinite the remainder R_n approaches zero. For let M be an upper bound for the absolute value of $f(z)$ along the circle C, R the radius of that circle, and r the absolute value of $x-a$. We have $|z-x| \geq R-r$, and therefore $|1/(z-x)| \leq 1/(R-r)$, when z describes the circle C. Hence the absolute value of R_n is less than

$$\frac{1}{2\pi}\left(\frac{r}{R}\right)^{n+1} \frac{M}{R-r} 2\,R\pi = \frac{MR}{R-r}\left(\frac{r}{R}\right)^{n+1},$$

and the factor $(r/R)^{n+1}$ approaches zero as n becomes infinite. From this it follows that $f(x)$ is equal to the convergent series

$$f(x) = J_0 + J_1(x-a) + \cdots + J_n(x-a)^n + \cdots.$$

[[80]] Now, if we put $x=a$ in the formulæ (12), (13), (14), the boundary Γ being here the circle C, we find [[80]]

$$J_0 = f(a), \qquad J_1 = f'(z), \qquad \cdots, \qquad J_n = \frac{f^{(n)}(a)}{n!}, \qquad \cdots.$$

The series obtained is therefore identical with the series (15); that is, with Taylor's series.

The circle C is a circle with center a, in the interior of which the function is analytic; it is clear that we would obtain the greatest circle satisfying that condition by taking for radius the distance from the point a to that singular point of $f(z)$ nearest a. This is also the circle of convergence for the series on the right.*

This important theorem brings out the identity of the two definitions for analytic functions which we have given (I, §197, 2d ed.; §191, 1st ed.; and II, §3). In fact, every power series represents an analytic function inside of its circle of convergence (§8); and, conversely, as we have just seen, every function analytic in a circle with the center a can be developed in a power series proceeding according to powers of $x-a$ and convergent inside of that circle. Let us also notice that a certain number of results previously established become now almost intuitive; for example, applying the theorem to the functions $\operatorname{Log}(1+z)$ and $(1+z)^m$, which are analytic inside of the circle of unit radius with the origin as center, we find again the formulæ of §§17 and 18.

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* This last conclusion requires some explanation on the nature of singular points, which will be given in the chapter devoted to analytic extension.

Let us now consider the quotient of two power series $f(x)/\phi(x)$, each convergent in a circle of radius R. If the series $\phi(x)$ does not vanish for $x = 0$, since it is continuous we can describe a circle of radius $\leq R$ in the whole interior of which it does not vanish. The function $f(x)/\phi(x)$ is therefore analytic in this circle of radius r and can therefore be developed in a power series in the neighborhood of the origin (I, §188, 2d ed.; §183, 1st ed.). In the same way, the theorem relative to the substitution of one series in another series can be proved, etc.

Note. Let $f(z)$ be an analytic function in the interior of a circle C with the center a and the radius r and continuous on the circle itself. The absolute value $|f(z)|$ of the function on the circle is a continuous function, the maximum value of which we shall indicate by $\mathcal{M}(r)$. On the other hand, the coefficient a_n of
[[81]] $(x-a)^n$ in the development of $f(z)$ is equal to $f^{(n)}(a)/n!$, that is, to [[81]]

$$\frac{1}{2\pi i}\int_{(C)} \frac{f(z)\,dz}{(z-a)^{n+1}};$$

we have, then,

$$(17) \qquad A_n = |a_n| < \frac{1}{2\pi}\frac{\mathcal{M}(r)}{r^{n+1}}2\pi r = \frac{\mathcal{M}(r)}{r^n},$$

so that $\mathcal{M}(r)$ is greater than all the products $A_n r^n$.* We could use $\mathcal{M}(r)$ instead of M in the expression for the dominant function (I, §186, 2d ed.; §181, 1st ed.).

36. Liouville's theorem. If the function $f(x)$ is analytic for every finite value of x, then Taylor's expansion is valid, whatever a may be, in the whole extent of the plane, and the function considered is called an *integral function.* From the expressions obtained for the coefficients we easily derive the following proposition, due to Liouville:

Every integral function whose absolute value is always less than a fixed number M is a constant.

For let us develop $f(x)$ in powers of $x-a$, and let a_n be the coefficient of $(x-a)^n$. It is clear that $\mathcal{M}(r)$ is less than M, whatever may be the radius r, and therefore $|a_n|$ is less than M/r^n. But the radius r can be taken just as large as we wish; we have, then, $a_n = 0$ if $n \geq 1$, and $f(x)$ reduces to a constant $f(a)$.

More generally, let $f(x)$ be an integral function such that the absolute value of $f(x)/x^m$ remains less than a fixed number M for values of x whose absolute value is greater than a positive number R; then *the function $f(x)$ is a polynomial of degree not greater than m.* For suppose we develop $f(x)$ in powers o fx, and let a_n be the coefficient of x^n. If the radius r of the circle C is greater than R, we have $\mathcal{M}(r) < Mr^m$, and consequently $|a_n| < Mr^{m-n}$. If $n > m$, we have then $a_n = 0$, since Mr^{m-n} can be made smaller than any given number by choosing r large enough.

37. Laurent's series. The reasoning by which Cauchy derived Taylor's series is capable of extended
[[82]] generalizations. Thus, let $f(z)$ be an analytic function in the ring-shaped region between the two concentric [[82]]
circles C, C' having the common center a. We shall show that *the value $f(x)$ of the function at any point x taken in that region is equal to the sum of two convergent series, one proceeding in positive powers of $x-a$, the other in positive powers of $1/(x-a)$.**

We can suppose, just as before, that the function $f(z)$ is analytic on the circles C, C' themselves. Let R, R' be the radii of these circles and r the absolute value of $x-a$; if C' is the interior circle, we have $R' < r < R$. About x as center let us describe a small circle γ lying entirely between C and C'. We have the equality

$$\int_{(C)} \frac{f(z)\,dz}{z-x} = \int_{(C')} \frac{f(z)\,dz}{z-x} + \int_{(\gamma)} \frac{f(z)\,dz}{z-x},$$

* The inequalities (17) are interesting, especially since they establish a relation between the order of magnitude of the coefficients of a power series and the order of magnitude of the function; $\mathcal{M}(r)$ is not, in general, however, the smallest number which satisfies these inequalities, as is seen at once when all the coefficients a_n are real and positive. These inequalities (17) can be established without making use of Cauchy's integral (MÉRAY, *Leçons nouvelles sur l'analyse infinitésimale*, Vol. I, p. 99).

* *Comptes rendus de l'Académie des Sciences*, Vol. XVII. See *Œuvres de Cauchy* 1st series, Vol. VIII, p. 115.

the integrals being taken in a suitable sense; the last integral, taken along γ, is equal to $2\pi i f(x)$, and we can write the preceding relation in the form

$$f(x) = \frac{1}{2\pi i}\int_{(C)} \frac{f(z)\,dz}{z-x} + \frac{1}{2\pi i}\int_{(C')} \frac{f(z)\,dz}{z-x} \tag{18}$$

where the integrals are all taken in the same sense.

Repeating the reasoning of §35, we find again that we have

$$\frac{1}{2\pi i}\int_{(C)} \frac{f(z)\,dz}{z-x} = J_0 + J_1(x-a) + \cdots + J_n(x-a)^n + \cdots, \tag{19}$$

where the coefficients $J_0,\ J_1,\ \cdots,\ J_n,\ \cdots$ are given by the formulæ (16). In order to develop the second integral in a series, let us notice that

$$\frac{1}{x-z} = \frac{1}{x-a}\left(\frac{1}{1-\frac{z-a}{x-a}}\right) = \frac{1}{x-a} + \frac{z-a}{+}\cdots + \frac{(z-a)^{n-1}}{(x-a)^n} + \frac{(z-a)^n}{(x-z)(x-a)^n},$$

and that the integral of the complementary term,

$$\frac{1}{2\pi i}\int_{(C')} \left(\frac{z-a}{x-a}\right)^n \frac{f(z)}{x-z}\,dz,$$

approaches zero when n increases indefinitely. In fact, if M' is the maximum of the absolute value of $f(z)$ along C', the absolute value of this integral is less than

$$\frac{1}{2\pi}\left(\frac{R'}{r}\right)^n \frac{M'}{r-R'} 2\pi R' = \frac{M'R'}{r-R'}\left(\frac{R'}{r}\right)^n,$$

[[83]] and the factor R'/r is less than unity. We have, then, also [[83]]

$$\frac{1}{2\pi i}\int_{(C')} \frac{f(z)\,dz}{x-z} = \frac{K_1}{x-a} + \frac{K_2}{(x-a)^2} + \cdots + \frac{K_n}{(x-a)^n} + \cdots, \tag{20}$$

where the coefficient K_n is equal to the definite integral

$$K_n = \frac{1}{2\pi i}\int_{(C')} (z-a)^{n-1} f(z)\,dz. \tag{21}$$

Adding the two developments (19) and (20), we obtain the proposed development of $f(x)$.

In the formulæ (16) and (21), which give the coefficients J_n and K_n, we can take the integrals along any circle Γ whatever lying between C and C' and having the point a for center, for the functions under the integral sign are analytic in the ring. Hence, if we agree to let the inex n vary from $-\infty$ to $+\infty$, we can write the development of $f(x)$ in the form

$$f(x) = \sum_{n=-\infty}^{+\infty} J_n(x-a)^n, \tag{22}$$

where the coefficient J_n, whatever the sign of n, is given by the formula

$$J_n = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{f(z)dz}{(z-a)^{n+1}}. \tag{23}$$

Example. The same function $f(x)$ can have developments which are entirely different, according to the region considered. Let us take, for example, a rational fraction $f(x)$, of which the denominator has only simple roots with different absolute values.

Let a, b, c, $\cdots$, l be these roots arranged in the order of increasing absolute values. Disregarding the integral part, which does not interest us here, we have

$$f(x) = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \cdots + \frac{L}{x-l}.$$

In the circle of radius a about the origin as center, each of the simple fractions can be developed in positive powers of x, and the development of $f(x)$ is identical with that given by Maclaurin's expansion

$$f(x) = -\left(\frac{A}{a} + \cdots + \frac{L}{l}\right) - \left(\frac{A}{a^2} + \cdots + \frac{L}{l^2}\right)x - \cdots - \left(\frac{A}{a^{n+1}} + \cdots + \frac{L}{l^{n+1}}\right)x^n - \cdots.$$

In the ring between the two circles of radii $|a|$ and $|b$— the fractions $1/(x-b)$, $1/(x-c)$, $\cdots$, $1/(x-l)$ can be developed in positive powers of x, but $1/(x-a)$ must be developed in positive powers of $1/x$, and we have

$$f(x) = -\left(\frac{B}{b} + \cdots + \frac{L}{l}\right) - \left(\frac{B}{b^2} + \cdots + \frac{L}{l^2}\right)x - \cdots - \left(\frac{B}{b^{n+1}} + \cdots + \frac{L}{l^{n+1}}\right)x^n - \cdots + \frac{A}{x} + \frac{Aa}{x^2} + \cdots + \frac{Aa^{n-1}}{x^n} + \cdots.$$

[[84]] In the next ring we shall have an analogous development, and so on. Finally, exterior to the circle of radius $|l|$, we shall have only positive powers of $1/x$: [[84]]

$$f(x) = \frac{A + \cdots + L}{x} + \frac{Aa + \cdots + Ll}{x^2} + \cdots + \frac{Aa^{n-1} + \cdots + Ll^{n-1}}{x^n} + \cdots.$$

38. Other series. The proofs of Taylor's series and of Laurent's series are based essentially on a particular development of the simple fraction $1/(z-x)$ when the point x remains inside or outside a fixed circle. Appell has shown that we can again generalize these formulæ by considering a function $f(x)$ analytic in the interior of a region A bounded by any number whatever of arcs of circles or of entire circumferences.* Let us consider, for example, a function $f(x)$ analytic in the curvilinear triangle PQR (Fig. 16) formed by the three arcs of circles PQ, QR, RP, belonging respectively to the three circumferences C, C', C''. Denoting by x any point within this curvilinear triangle, we have

$$f(x) = \frac{1}{2\pi i}\int_{(PQ)} \frac{f(z)\,dz}{z-x} + \frac{1}{2\pi i}\int_{(QR)} \frac{f(z)\,dz}{z-x} + \frac{1}{2\pi i}\int_{(RP)} \frac{f(z)\,dz}{z-x}. \tag{24}$$

Along the arc PQ we can write

$$\frac{1}{z-x} = \frac{1}{z-a} + \frac{x-a}{(z-a)^2} + \cdots + \frac{(x-a)^n}{(z-a)^{n+1}} + \frac{1}{z-x}\left(\frac{x-a}{z-a}\right)^{n+1},$$

where a is the center of C; but when z describes the arc PQ, the absolute value of $(x-a)/(z-a)$ is less than unity, and therefore the absolute value of the integral

$$\frac{1}{2\pi i}\int_{(PQ)} \frac{f(z)}{z-x}\left(\frac{x-a}{z-a}\right)^{n+1} dz$$

approaches zero as n becomes infinite. We have, therefore,

$$\frac{1}{2\pi i}\int_{(PQ)} \frac{f(z)\,dz}{z-x} = J_0 + J_1(x-a) + \cdots + J_n(x-a)^n + \cdots, \tag{α}$$

[[85]] where the coefficients are constants whose expressions it would be easy to write out. Similarly, along the arc QR we can write [[85]]

$$\frac{1}{x-z} = \frac{1}{x-b} + \frac{z-b}{(x-b)^2} + \cdots + \frac{(z-b)^{n-1}}{(x-b)^n} + \frac{1}{x-z}\left(\frac{z-b}{x-b}\right)^n,$$

where b is the center of C'. Since the absolute value of $(z-b)^n/(x-b)^n$ approaches zero as n becomes infinite, we can deduce from the preceding equation a development for the second integral of the form

$$\frac{1}{2\pi i}\int_{(QR)} \frac{f(z)\,dz}{z-x} = \frac{K_1}{x-b} + \frac{K_2}{(x-b)^2} + \cdots + \frac{K_n}{(x-b)^n} + \cdots. \tag{β}$$

* *Acta mathematica*, Vol. I, p. 145.

Similarly, we find

$$(\gamma)\qquad \frac{1}{2\pi i}\int_{(RP)} \frac{f(z)\,dz}{z-x} = \frac{L_1}{x-c} + \frac{L_2}{(x-c)^2} + \cdots + \frac{L_n}{(x-c)^n} + \cdots,$$

where c is the center of the circle C''. Adding the three expressions (α), (β), (γ), we obtain for $f(x)$ the sum of three series, proceeding respectively according to positive powers of $x-a$, of $1/(x-b)$, and of $1/(x-c)$. It is clear that we can transform this sum into a series of which all the terms are rational functions of x, for example, by uniting all the terms of the same degree in $x-a$, $1/(x-b)$, $1/(x-c)$. The preceding reasoning applies whatever may be the number of arcs of circles.

It is seen in the preceding example that the three series, (α), (β), (γ), are still convergent when the point x is inside the triangle $P'Q'R'$, and the sum of these three series is again equal to the integral

$$\int \frac{f(z)\,dz}{z-x}$$

taken along the boundary of the triangle PQR in the positive sense. Now, when the point x is in the triangle $P'Q'R'$, the function $f(z)/(z-x)$ is analytic in the interior of the triangle PQR, and the preceding integral is therefore zero. Hence we obtain in this way a series of rational fractions which is convergent when x is within one of the two triangles PQR, $P'Q'R'$, and for which *the sum is equal to $f(x)$ or to zero, according as the point x is in the triangle PQR or in the triangle $P'Q'R'$.*

Painlevé has obtained more general results along the same lines.* Let us consider, in order to limit ourselves to a very simple case, a convex closed curve Γ having a tangent which changes continuously and a radius of curvature which remains under a certain upper bound. It is easy to see that we can associate with each point M of Γ a circle C tangent to Γ at that point and inclosing that curve entirely in its interior, and this may be done in such a way that the center of the circle moves in a continuous manner with M. Let $f(z)$ be a function analytic in the interior of the boundary Γ and continuous on the boundary itself. Then, in the fundamental formula

$$f(x) = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{f(z)\,dz}{z-x},$$

[[86]] where x is an interior point to Γ, we can write [[86]]

$$\frac{1}{z-x} = \frac{1}{z-a} + \frac{x-a}{(z-a)^2} + \cdots + \frac{(x-a)^n}{(z-a)^{n+1}} + \frac{1}{z-x}\left(\frac{x-a}{z-a}\right)^{n+1},$$

where a denotes the center of the circle C which corresponds to the point z of the boundary; a is no longer constant, as in the case already examined, but it is a continuous function of z when the point M describes the curve Γ. Nevertheless, the absolute value of $(x-a)/(z-a)$, which is a continuous function of z, remains less than a fixed number ρ less than unity, since it cannot reach the value unity, and therefore the integral of the last term approaches zero as n becomes infinite. Hence we have

$$(25)\qquad f(x) = \frac{1}{2\pi i}\sum_{n=0}^{+\infty}\int_{(\Gamma)} \frac{(x-a)^n}{(z-a)^{n+1}} f(z)\,dz,$$

and it is clear that the general term of this series is a polynomial $P_n(x)$ of degree not greater than n. *The function $f(x)$ is then developable in a series of polynomials in the interior of the boundary Γ.*

The theory of conformal transformations enables us to obtain another kind of series for the development of analytic functions. Let $f(x)$ be an analytic function in the interior of the region A, which may extend to infinity. Suppose that we know how to represent the region A conformally on the region inclosed by a circle C such that to a point of the region A corresponds one and only one point of the circle, and conversely; let $u = \phi(z)$ be the analytic function which establishes a correspondence between the region A and the circle C having the point $u = 0$ for center in the u-plane. When the variable u describes this circle, the corresponding value of z is an analytic function of u. The same is true of $f(z)$, which can therefore be developed in a convergent series of powers of u, or of $\phi(z)$, when the variable z remains in the interior of A.

Suppose, for example, that the region A consists of the infinite strip included between the two parallels to the axis of reals $y = \pm a$. We have seen (§22) that by putting $u = (e^{\pi z/2a} - 1)/(e^{\pi z/2a} + 1)$ this strip is made to correspond to a circle of unit radius having its center at the point $u = 0$. Every function analytic in this strip can therefore be developed in this strip in a convergent series of the following form:

$$f(z) = \sum_{n=0}^{+\infty} A_n \left(\frac{e^{\frac{\pi z}{2a}} - 1}{e^{\frac{\pi z}{2a}} + 1}\right)^n.$$

39. Series of analytic functions. The sum of a uniformly convergent series whose terms are analytic functions of z is a continuous function of z, but we could not say without further proof that that

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* *Sur les lignes singulières des fonctions analytiques* (*Annales de la Faculté de Toulouse,* 1888).

sum is also an analytic function. It must be proved that the sum has a unique derivative at every point, and this is easy to do by means of Cauchy's integral.

Let us first notice that a uniformly convergent series whose terms are continuous functions of a complex
[[87]] variable z can be integrated term by term, as in the case of a real variable. The proof given in the case of [[87]]
the real variable (I, §114, 2d ed.; §174, 1st ed.) applies here without change, provided the path of integration has a finite length.

The theorem which we wish to prove is evidently included in the following more general proposition:

Let

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots \tag{26}$$

be a series all of whose terms are analytic functions in a region A bounded by a closed curve Γ and continuous on the boundary. If the series (26) is uniformly convergent on Γ, it is convergent in every point of A, and its sum is an analytic function $F(z)$ whose pth *derivative is represented by the series formed by the* pth *derivatives of the terms of the series (26).*

Let $\phi(z)$ be the sum of (26) in a point of Γ; $\phi(z)$ is a continuous function of z along the boundary, and we have seen (§33, *Note*) that the definite integral

$$F(x) = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{\phi(z)\,dz}{z-x} = \frac{1}{2\pi i}\int_{(\Gamma)} \frac{\sum_{\nu=1}^{+\infty} f_\nu(z)}{z-x}\,dz, \tag{27}$$

where x is any point of A, represents an analytic function in the region A, whose pth derivative is the expression

$$F^{(p)}(x) = \frac{p!}{2\pi i}\int_{(\Gamma)} \frac{\phi(z)\,dz}{(z-x)^{p+1}} = \frac{p!}{2\pi i}\int_{(\Gamma)} \frac{\sum_{\nu=1}^{+\infty} f_\nu(z)}{(z-x)^{p+1}}\,dz. \tag{28}$$

Since the series (26) is uniformly convergent on Γ, the same thing is true of the series obtained by dividing each of its terms by $z - x$, and we can write

$$F(x) = \sum_{\nu=1}^{+\infty} \frac{1}{2\pi i}\int_{(\Gamma)} \frac{f_\nu(z)\,dz}{z-x};$$

or again, since $f_\nu(z)$ is an analytic function in the interior of Γ, we have, by formula (12),

$$F(x) = f_1(x) + f_2(x) + \cdots + f_\nu(x) + \cdots.$$

Similarly, the expression (28) can be written in the form

$$F^{(p)}(x) = f_1^{(p)}(x) + \cdots + f_\nu^{(p)}(x) + \cdots.$$

Hence, if the series (26) is uniformly convergent in a region A of the plane, x being any point of that
[[88]] region, it suffices to apply the preceding theorem to a closed curve Γ lying in A and surrounding the point [[88]]
x. This leads to the following proposition:

Every series uniformly convergent in a region A of the plane, whose terms are all analytic functions in A, represents an analytic function $F(z)$ in the same region. The pth *derivative of $F(z)$ is equal to the series obtained by differentiating p times each term of the series which represents $F(z)$.*

40. Poles. Every function analytic in a circle with the center a is equal, in the interior of that circle, to the sum of a power series

$$f(z) = A_0 + A_1(z-a) + \cdots + A_m(z-a)^m + \cdots. \tag{29}$$

We shall say, for brevity, that the function is *regular* at the point a, or that a is an *ordinary point* for the given function. We shall call the interior of a circle C, described about a as a center with the radius ρ, the *neighborhood* of the point a, when the formula (29) is applicable. It is, moreover, not necessary that this shall be the largest circle in the interior of which the formula (29) is true; the radius ρ of the neighborhood will often be defined by some other particular property.

If the first coefficient A_0 is zero, we have $f(a) = 0$, and the point a is a zero of the function $f(z)$. The order of a zero is defined in the same way as for polynonmials; if the development of $f(z)$ commences with a term of degree m in $z - a$,

$$f(z) = A_m(z-a)^m + A_{m+1}(z-a)^{m+1} + \cdots, \qquad (m > 0),$$

where A_m is not zero, we have

$$f(a) = 0, \quad f'(a) = 0, \quad ,\cdots, \quad f^{(m-1)}(a) = 0, \quad f^{(m)}(a) \neq 0,$$

and the point a is said to be a *zero of order m*. We can also write the preceding formula in the form

$$f(z) = (z-a)^m \phi(z),$$

$\phi(z)$ being a power series which does not vanish when $z = a$. Since this series is a continuous function of z, we can choose the radius ρ of the neighborhood so small that $\phi(z)$ does not vanish in that neighborhood, and we see that the function $f(z)$ will not have any other zero than the point a in the interior of that neighborhood. *The zeros of an analytic function are therefore isolated points.*

Every point which is not an ordinary point for a single-valued function $f(z)$ is said to be a *singular*
[[89]] *point.* A singular point a of the function $f(z)$ is a *pole* if that point is an ordinary point for the reciprocal [[89]]
function $1/f(z)$. The development of $1/f(z)$ in powers of $z-a$ cannot contain a constant term, for the point a would then be an ordinary point for the function $f(z)$. Let us suppose that the development commences with a term of degree m in $z - a$,

$$\frac{1}{f(z)} = (z-a)^m \phi(z), \tag{30}$$

where $\phi(z)$ denotes a regular function in the neighborhood of the point a which is not zero when $z = a$. From this we derive

$$f(z) = \frac{1}{(z-a)^m}\frac{1}{\phi(z)} = \frac{\psi(z)}{(z-a)^m}, \tag{31}$$

where $\psi(z)$ denotes a regular function in the neighborhood of the point a which is not zero when $z = a$. This formula can be written in the equivalent form

$$f(z) = \frac{B_m}{(z-a)^m} + \frac{B_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{B_1}{z-a} + P(z-a), \tag{31$'$}$$

where we denote by $P(z-a)$, as we shall often do hereafter, a regular function for $z = a$, and by B_m, B_{m-1}, $\cdots$, B_1 certain constants. Some of the coefficients B_1, B_2, $\cdots$, B_{m-1} may be zero, but the coefficient B_m is surely different from zero. The integer m is called the *order of the pole.* It is seen that a pole of order m of $f(z)$ is a zero of order m of $1/f(z)$, and conversely.

In the neighborhood of a pole a the development of $f(z)$ is composed of a regular part $P(z-a)$ and of a polynomial in $1/(z-a)$; this polynomial is called the *principal part* of $f(z)$ in the neighborhood of the pole. *When the absolute value of $z - a$ approaches zero, the absolute value of $f(z)$ becomes infinite in whatever way the point z approaches the pole.* In fact, since the function $\psi(z)$ is not zero for $z = a$, suppose the radius of the neighborhood so small that the absolute value of $\psi(z)$ remains greater than a positive number M in this neighborhood. Denoting by r the absolute value of $z - a$, we have $|f(z)| > M/r^m$, and therefore $|f(z)|$ becomes infinite when r approaches zero. Since the function $\psi(z)$ is regular for $z = a$, there exists a circle C with the center a in the interior of which $\psi(z)$ is analytic. The quotient $\psi(z)/(z-a)^m$ is an analytic function for all the points of this circle except for the point a itself. In the neighborhood of a pole a, the function $f(z)$ has therefore no other singular point than the pole itself; in other words, *poles are isolated singular points.*

[[90]] **41. Functions analytic except for poles.** Every function which is analytic at all the points of a region A, except only for singular points that are poles, is said to be *analytic except for poles in that region.** A function analytic in the whole plane except for poles may have an infinite number of poles, but it can have only a finite number in any finite region of the plane. The proof depends on a general theorem, which we must now recall; *If in a finite region A of the plane there exist an infintie number of points possessing a particular property, there exists at least one limit point in the region A or on its boundary.* (We mean by *limit point* a point in every neighborhood of which there exist an infinite number of points possessing the given property.) This proposition is proved by the process of successive subdivisions that we have employed so often. For brevity, let us indicate by (E) the assemblage of points considered, and let us suppose that the region A is divided into squares, or portions of squares, by parallels to the axes Ox, Oy. There will be at least one region A_1 containing an infinite number of points of the assemblage (E). By subdividing the region A_1 in the same way, and by continuing this process indefinitely, we can form an infinite sequence of regions $A_1, A_2, \cdots, A_n, \cdots$ that become smaller and smaller, each of which is contained in the preceding and contains an infinite number of the points of the assemblage. All the points of A_n approach a limit point Z lying in the interior of or on the boundary of A. The point Z is necessarily a limit point of (E), since there are always an infintie number of points of (E) in the interior of a circle having Z for center, however small the radius of that circle may be. [[90]]

Let us now suppose that the function $f(z)$ is analytic except for poles in the interior of a finite region A and also on the boundary Γ of that region. If it has an infintie number of poles in the region, it will have, by the preceding theorem, at least one point Z situated in A or on Γ, in every neighborhood of which it will have an infinite number of poles. hence the point Z can be neither a pole nor an ordinary point. It is seen in the same way that the function $f(z)$ can have only a finite number of zeros in the same region. It follows that we can state the following theorem:

Every function analytic except for poles in a finite region A and on its boundary has in that region only a finite number of zeros and only a finite number of poles.

[[91]] In the neighborhood of any point a, a function $f(z)$ analytic except for poles can be put in the form [[91]]

$$f(z) = (z-a)^{\mu}\phi(z), \tag{32}$$

where $\phi(z)$ is a regular function not zero for $z = a$. The exponent μ is called the *order* of $f(z)$ at the point a. The order is zero if the point a is neither a pole nor a zero for $f(z)$; it is equal to m if the point a is a zero of order m for $f(z)$, and to $-n$ if a is a pole of order n for $f(z)$.

42. Essentially singular points. Every singular point of a single-valued analytic function, which is not a pole, is called an *essentially singular point.* An essentially singular point a is isolated if it is possible to describe about a as a center a circle C in the interior of which the function $f(z)$ has no other singular point than the point a itself; we shall limit ourselves for the moment to such points.

Laurent's theorem furnishes at once a development of the function $f(z)$ that holds in the neighborhood of an essentially singular point. Let C be a circle, with the center a, in the interior of which the function $f(z)$ has no other singular point than a; also let c be a circle concentric with and interior to C. In the circular ring included between the two circles C and c the function $f(z)$ is analytic and is therefore equal to the sum of a series of positive and negative powers of $z - a$,

$$f(z) = \sum_{m=-\infty}^{+\infty} A_m (z-a)^m. \tag{33}$$

This development holds true for all the points interior to the circle C except the point a, for we can always take the radius of the circle c less than $|z - a|$ for any point z whatever that is different from a and lies in

* Such functions are said by some writers to be *meromorphic.* – Trans.

C. Moreover, the coefficients A_m do not depend on this radius (§37). The development (33) contains first a part regular at the point a, say $P(z-a)$, formed by the terms with positive exponents, and then a series of terms in powers of $1/(z-a)$,

$$\frac{A_{-1}}{z-a} + \frac{A_{-2}}{(z-a)^2} + \cdots + \frac{A_{-m}}{(z-a)^m} + \cdots. \tag{34}$$

This is the *principal part* of $f(z)$ in the neighborhood of the singular point. This principal part does not
[[92]] reduce to a polynomial in $(z-a)^{-1}$, for the point $z = a$ would then be a pole, contrary to the hypothesis.* [[92]]
It is an *integral transcendental function* of $1/(z-a)$. In fact, let r be any positive number less than the radius of the circle C; the coefficient A_{-m} of the series (34) is given by the expression (§37)

$$A_{-m} = \frac{1}{2\pi i} \int_{(C')} (z-a)^{m-1} f(z)\,dz,$$

the integral being taken along the circle C' with the center a and the radius r. We have, then,

$$|A_{-m}| < \mathcal{M}(r) r^m, \tag{35}$$

where $\mathcal{M}(r)$ denotes the maximum of the absolute value of $f(z)$ along the circle C'. The series is then convergent, provided that $|z-a|$ is greater than r, and since r is a number which we may suppose as small as we wish, the series (34) is convergent for every value of z different from a, and we can write

$$f(z) = P(z-a) + G\left(\frac{1}{z-a}\right),$$

where $P(z-a)$ is a regular function at the point a, and $G[1/(z-a)]$ an integral transcendental function† of $1/(z-a)$.

When the absolute value of $z-a$ approaches zero, the value of $f(z)$ does not approach any definite limit. more precisely, *if a circle C is described with the point a as a center and with an arbitrary radius ρ, there always exists in the interior of this circle points z for which $f(z)$ differs as little as we please from any number given in advance* (Weierstrass).

Let us first prove that, given any two positive numbers ρ and M, there exist values of z for which both the inequalities, $|z-a| < \rho$, $|f(z)| > M$, hold. For, if the absolute value of $f(z)$ were at most equal to M when we have $|z-a| < \rho$, $\mathcal{M}(r)$ would be less than or equal to M for r, ρ, and, from the inequality (35), all the coefficients A_{-m} would be zero, for the product $\mathcal{M}(r)r^m \leq Mr^m$ would approach zero with r.

Let us consider now any value A whatever. If the equation $f(z) = A$ has roots within the circle C,
[[93]] however small the radius ρ may be, the theorem is proved. If the equation $f(z) = A$ does not have an infinite [[93]]
number of roots in the neighborhood of the point a, we can take the radius ρ so small that in the interior of the circle C with the radius ρ and the center a this equation does not have any roots. The function $\phi(z) = 1/[f(z) - A]$ is then analytic for every point z within C except for the point a; this point a cannot be anything but an essentially singular point for $\phi(z)$, for otherwise the point would be either a pole or an ordinary point for $f(z)$. Therefore, from what we have just proved, there exist values of z in the interior of the circle C for which we have

$$|\phi(z)| > \frac{1}{\epsilon} \quad \text{or} \quad |f(z) - A| < \epsilon,$$

however small the positive number ϵ may be.

* To avoid overlooking any hypothesis, it would be necessary to examine also the case in which the development of $f(z)$ in the interior of C contains only positive powers of $z-a$, the value $f(a)$ of the function at the point a being different from the term independent of $z-a$ in the series. The point $z=a$ would be a *point of discontinuity* for $f(z)$. We shall disregard this kind of singularity, which is of an entirely artificial character (see below, Chapter IV).

† We shall frequently denote an integral function of x by $G(x)$.

This property sharply distinguishes poles from essentially singular points. While the absolute value of the function $f(z)$ becomes infinite in the neighborhood of a pole, the value of $f(z)$ is completely indeterminate for an essentially singular point.

Picard* has demonstrated a more precise proposition by showing that every equation $f(z) = A$ has an infintie number of roots in the neighborhood of an essentially singular point, there being no exception except for, at most, one particular value of A.

Example. The point $z = 0$ is an essentially singular point for the function

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots + \frac{1}{n!}\frac{1}{z^n} + \cdots.$$

It is easy to prove that the equation $e^{1/z} = A$ has an infinite number of roots with absolute values less than ρ, however small ρ may be, provided that A is not zero. Setting $A = r(\cos\theta + i\sin\theta)$, we derive from the preceding equation

$$\frac{1}{z} = \log r + i(\theta + 2k\pi).$$

We shall have $|z| < \rho$, provided that

$$(\log r)^2 + (\theta + 2k\pi)^2 \geq \frac{1}{\rho^2}.$$

There are evidently an infinite number of values of the integer k which satisfy this condition. In this example there is one exceptional value of A, that is, $A = 0$. But it may also happen that there are no exceptional values; such is the case, for example, for the function $\sin(1/z)$, near $z = 0$.

[[94]] **43. Residues.** Let a be a pole or an isoalted essentially singular point of a function $f(z)$. Let us [[94]] consider the question of evaluating the integral $\int f(z)\,dz$ along the circle C drawn in the neighborhood of the point a with the center a. The regular part $P(z-a)$ gives zero in the integration. As for the principal part $G[1/(z-a)]$, we can integrate it term by term, for, even though the point a is an essentially singular point, this series is uniformly convergent. The integral of the general term

$$\int_{(C)} \frac{A_{-m}\,dz}{(z-a)^m}$$

is zero if the exponent m is greater than unity, for the primitive function $-A_{-m}/[(m-1)(z-a)^{m-1}]$ takes on again its original value after the variable has described a closed path. If, on the contrary, $m = 1$, the definite integral $A_{-1}\int dz/(z-a)$ has the value $2\pi i A_{-1}$, as was shown by the previous evaluation made in §34. We have then the result

$$2\pi i A_{-1} = \int_{(C)} f(z)\,dz,$$

which is essentially only a particular case of the formula (23) for the coefficients of the Laurent development. The coefficient A_{-1} is called the *residue* of the function $f(z)$ with respect to the singular point a.

Let us consider now a function $f(z)$ continuous on a closed boundary curve Γ and having in the interior of that curve Γ only a finite number of singular points a, b, c, $\cdots$, l. Let A, B, C, $\cdots$, L be the corresponding residues; if we surround each of these singular points with a circle of very small radius, the integral $\int f(z)\,dz$, taken along Γ in the positive sense, is equal to the sum of the integrals taken along the small curves in the same sense, and we have the very important formula

$$\int_{(\Gamma)} f(z)\,dz = 2\pi i(A + B + C + \cdots + L), \tag{36}$$

which says that *the integral* $\int f(z)\,dz$*, taken along* Γ *in the positive sense, is equal to the product of* $2\pi i$ *and the sum of the residues with respect to the singular points of* $f(z)$ *within the curve* Γ.

It is clear that the theorem is also applicable to boundaries Γ composed of several distinct closed curves. The importance of residues is now evident, and it is useful to know how to calculate them rapidly. If a point

* *Annales de l'École Normale supérieure*, 1880.

a is a pole of order m for $f(z)$, the product $(z-a)^m f(z)$ is regular at the point a, and the residue of $f(z)$
[[95]] is evidently the coefficient of $(z-a)^{m-1}$ in the development of that product. The rule becomes simple in [[95]]
the case of a simple pole; the residue is then equal to the limit of the product $(z-a)f(z)$ for $z=a$. Quite frequently the function $f(z)$ appears under the form

$$f(z) = \frac{P(z)}{Q(z)},$$

where the functions $P(z)$ and $Q(z)$ are regular for $z=a$, and $P(a)$ is different from zero, while a is a simple zero for $Q(z)$. Let $Q(z) = (z-a)R(z)$; then the residue is equal to the quotient $P(a)/R(a)$, or again, as it is easy to show, to $P(a)/Q'(a)$.

III. APPLICATIONS OF THE GENERAL THEOREMS

The applications of the last theorem are innumerable. We shall now give some of them which are related particularly to the evaluation of definite integrals and to the theory of equations.

44. Introductory remarks. Let $f(z)$ be a function such that the product $(z-a)f(z)$ approaches zero with $|z-a|$. The integral of this function along a circle γ, with the center a and the radius ρ, approaches zero with the radius of that circle. Indeed, we can write

$$\int_{(\gamma)} f(z)\,dz = \int_{(\gamma)} (z-a)f(z)\frac{dz}{z-a}.$$

If η is the maximum of the absolute value of $(z-a)f(z)$ along the circle γ, the absolute value of the integral is less than $2\pi\eta$, and consequently approaches zero, since η itself is infinitesimal with ρ. We could show in the same way that, when the product $(z-a)f(z)$ approaches zero as the absolute value of $z-a$ becomes infinite, the integral $\int_{(C)} f(z)\,dz$, taken along a circle C with the center a, approaches zero as the radius of the circle becomes infinite. These statements are still true if, instead of integrating along the entire circumference, we integrate along only a part of it, provided that the product $(z-a)f(z)$ approaches zero along that part.

Frequently we have to find an upper bound for the absolute value of a definite integral of the form $\int_a^b f(x)\,dx$, taken along the axis of reals. Let us suppose for definiteness $a<b$. We have seen above (§25) that the absolute value of that integral is at most equal to the integral $\int_a^b |f(x)|\,dx$, and, consequently, is less than $M(b-a)$ if M is an upper bound of the absolute value of $f(x)$.

[[96]] **45. Evaluation of elementary definite integrals.** The definite integral $\int_{-\infty}^{+\infty} F(x)\,dx$, taken along [[96]]
the real axis, where $F(x)$ is a rational function, has a sense, provided that the denominator does not vanish for any real value of x and that the degree of the numerator is less than the degree of the denominator by at least two units. With the origin as center let us describe a circle C with a radius R large enough to include all the roots of the denominator of $F(z)$, and let us consider a path of integration formed by the diameter BA, traced along the real axis, and the semicircumference C', lying above the real axis. The only singular points of $F(z)$ lying in the interior of this path are poles, which come from the roots of the denominator of $F(z)$ for which the coefficient of i is positive. Indicating by $\sum R_k$ the sum of the residues relative to these poles, we can then write

$$\int_{-R}^{+R} F(z)\,dz + \int_{(C')} F(z)\,dz = 2\pi i \sum R_k.$$

As the radius R becomes infinite the integral along C' approaches zero, since the product $zF(z)$ is zero for z infinite; and, taking the limit, we obtain

$$\int_{-\infty}^{+\infty} F(x)\,dx = 2\pi i \sum R_k.$$

We easily reduce to the preceding case the definite integrals

$$\int_0^{2\pi} F(\sin x, \cos x)\,dx,$$

where F is a rational function of $\sin x$ And $\cos x$ that does not become infinite for any real value of x, and where the integral is to be taken along the axis of reals. Let us first notice that we do not change the value of this integral by taking for the limits x_0 and $x_0 + 2\pi$, where x_0 is any real number whatever. It follows that we can take for the limits $-\pi$ and $+\pi$, for example. Now the classic change of variable $\tan(x/2) = t$ reduces the given integral to the integral of a rational function of t taken between the limits $-\infty$ and $+\infty$, for $\tan(x/2)$ increases from $-\infty$ to $+\infty$ when x increases from $-\pi$ to $+\pi$.

We can alos roceed in another way. By putting $e^{xi} = z$ we have $dx = dz/iz$, and Euler's formulæ give

$$\cos x = \frac{z^2+1}{2z}, \qquad \sin x = \frac{z^2-1}{2iz},$$

[[97]] so that the given integral takes the form [[97]]

$$\int F\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right)\frac{dz}{iz}.$$

As for the new path of integration, when x increases from 0 to 2π the variable z describes in the positive sense the circle ofunit radius about the origin as center. It will suffice, then, to calculate the residues of the new rational function of z with respect to the poles whose absolute values are less than unity.

Let us take for example the integral $\int_0^{2\pi} \operatorname{ctn}[(x-a-bi)/2\,dx$, which has a finite value if b is not zero. We have

$$\operatorname{ctn}\left(\frac{x-a-bi}{2}\right) = i\frac{e^{i\left(\frac{x-a-bi}{2}\right)} + e^{-i\left(\frac{x-a-bi}{2}\right)}}{e^{i\left(\frac{x-a-bi}{2}\right)} - e^{-i\left(\frac{x-a-bi}{2}\right)}},$$

or

$$\operatorname{ctn}\left(\frac{x-a-bi}{2}\right) = i\frac{e^{ix}+e^{-b+ai}}{e^{ix}-e^{-b+ai}}.$$

Hence the change of variable $e^{xi} = z$ leads to the integral

$$\int_{(C)} \frac{z+e^{-b+ai}}{z-e^{-b+ai}}\,\frac{dz}{z}.$$

The function to be integrated has two simple poles

$$z = 0, \qquad z = e^{-b+ai},$$

and the corresponding residues are -1 and $+2$. If b is positive, the two poles are in the interior of the path of integration, and the integral is equal to $2\pi i$; if b is negative, the pole $z = 0$ is the only one within the path, and the integral is equal to $-2\pi i$. The proposed integral is therefore equal to $\pm 2\pi i$, according as b is positive or negative. We shall now give some examples which are less elementary.

46. Various definite integrals. *Example* 1. The function $e^{imz}/(1+z^2)$ has the two poles $+i$ and $-i$, with the residues $e^{-m}/2i$ and $-e^m/2i$. Let us suppose for definiteness that m is positive, and let us consider the boundary formed by a large semicircle of radius R about the origin as center and above the real axis, and by the diameter which falls along the axis of reals. In the interior of this boundary the function $e^{miz}/(1+z^2)$ has the single pole $z = i$, and the integral taken along the total boundary is equal to πe^{-m}. Now the integral along the semicircle approaches zero as the radius R becomes infinite, for the absolute value of the product $ze^{imz}/(1+z^2)$ along that curve approaches zero. Indeed, if we replace z by $R(\cos\theta + i\sin\theta)$, we have

$$e^{miz} = e^{-mR\sin\theta + imR\cos\theta},$$

[[98]] and the absolute value $e^{-mR\sin\theta}$ remains less than unity when θ varies from 0 to π. As for the absolute value of the factor [[98]] $z/(1+z^2)$, it approaches zero as z becoes infinite. We have, then, in the limit

$$\int_{-\infty}^{+\infty} \frac{e^{mix}}{1+x^2}\,dx = \pi e^{-m}.$$

If we replace e^{mix} by $\cos mx + i\sin mx$, the coefficient of i on the left-hand side is evidently zero, for the elements of the integral cancel out in pairs. Since we have also $\cos(-mx) = \cos mx$, we can write the preceding formula in the form

$$\int_0^{+\infty} \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{im}. \tag{37}$$

Example 2. The function e^{iz}/z is analytic in the interior of the boundary $ABMB'A'NA$ (Fig. 17) formed by the two semicircles BMB', $A'NA$, described about the origin as center with the radii R and r, and the straight lines AB, $B'A'$.

We have, then, the relation

$$\int_r^R \frac{e^{ix}}{x}\,dx + \int_{(BMB')} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{(A'NA)} \frac{e^{iz}}{z}\,dz = 0,$$

which we can write also in the form

$$\int_r^R \frac{e^{ix}-e^{-ix}}{x}\,dx + \int_{(BMB')} \frac{e^{iz}}{z}\,dz + \int_{(A'NA)} \frac{e^{iz}}{z}\,dz = 0.$$

When r approaches zero, the last integral approaches $-\pi i$; we have, in fact,

$$\frac{e^{iz}}{z} = \frac{1}{z} + P(z),$$

where $P(z)$ is a regular function at the origin, so that

$$\int_{(A'NA)} \frac{e^{iz}}{z}\,dz = \int_{(A'NA)} P(z)\,dz + \int_{(A'NA)} \frac{dz}{z}.$$

The integral of the regular part $P(z)$ becomes infinitesimal with the length of the path of integration; as for the last integral, it is equal to the variation of $\operatorname{Log}(z)$ along $A'NA$, that is, to $-\pi i$.

The integral along BMB' approaches zero as R becomes infinite. For if we put $z = R(\cos\theta + i\sin\theta)$, we find

$$\int_{(BMB')} \frac{e^{iz}}{z}\,dz = i\int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\,d\theta,$$

and the absolute value of this integral is less than

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta.$$

[[99]] When θ increases from 0 to $\pi/2$, the quotient $\sin\theta/\theta$ decreases from 1 to $2/\pi$, and we have [[99]]

$$R\sin\theta > \frac{2}{\pi}R\theta;$$

hence

$$e^{-R\sin\theta} < e^{-\frac{2R\theta}{\pi}},$$
$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta < \int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}\,d\theta = -\frac{\pi}{2R}\left[e^{-\frac{2R\theta}{\pi}}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2R}(1-e^{-R});$$

which establishes the proposition stated above.

Passing to the limit, we have, then (see I, §100, 2d ed.),

$$\int_0^{+\infty} \frac{e^{ix}-e^{-ix}}{x}\,dx = \pi i,$$

or

$$\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Example 3. The integral of the integral transcendental function e^{-z^2} along the boundary $OABO$ formed by the two radii OA and OB, making an angle of 45°, and by the arc of a circle AB (Fig. 18), is equal to zero, and this fact can be expressed as follows:

$$\int_0^R e^{-x^2}\,dx + \int_{(AB)} e^{-z^2}\,dz = \int_{(OB)} e^{-z^2}\,dz.$$

When the radius R of the circle to which the arc AB belongs becomes infinite, the integral along the arc AB approaches zero. In fact, if we put $z = R[\cos(\pi/2) + i\sin(\phi/2)]$, that integral becomes

$$\frac{iR}{2}\int_0^{\frac{\pi}{2}} e^{-R^2(\cos\phi + i\sin\phi)} e^{\frac{i\phi}{2}}\,d\phi,$$

and its absolute value is less than the integral

$$\frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\cos\phi}\,d\phi.$$

As in the previous example, we have

$$\frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\cos\phi}\,d\phi = \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\sin\phi}\,d\phi < \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-\frac{2R^2\phi}{\pi}}\,d\phi.$$

The last integral has the value

$$-\frac{\pi}{4R}\left[e^{-\frac{2R^2\phi}{\pi}}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4R}(1 - e^{-R^2})$$

and approaches zero when R becomes infinite.

[[100]] Along the radius OB we can put $z = \rho[\cos(\pi/4) + i\sin(\pi/4)]$, which gives $e^{-z^2} = e^{-i\rho^2}$, and as R becomes infinite we [[100]]
have at the limit (see I, §135, 2d ed.; §134, 1st ed.)

$$\int_0^{+\infty} e^{-i\rho^2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)d\rho = \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

or, again,

$$\int_0^{+\infty} e^{-i\rho^2}\,d\rho = \frac{\sqrt{\pi}}{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right).$$

Equating the real parts and the coefficients of i, we obtain the values of Fresnel's integrals,

$$(38)\qquad \int_0^{+\infty}\cos\rho^2\,d\rho = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \qquad \int_0^{+\infty}\sin\rho^2\,d\rho = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

47. Evaluation of $\Gamma(p)\Gamma(1-p)$. The definite integral

$$\int_0^{+\infty}\frac{x^{p-1}\,dx}{1+x},$$

where the variable x and the exponent p are real, has a finite value, provided that p is positive and less than one; it is equal to the product $\Gamma(p)\Gamma(1-p)$.* In order to evaluate this integral, let us consider the function $z^{p-1}/(1+z)$, which has a pole at the point $z = -1$ and a branch point at the point $z = 0$. Let us consider the boundary $abmb'a'na$ (Fig. 19) formed by the two circles C and C', described about the origin with the radii r and ρ respectively, and the two straight lines ab and $a'b$, lying as near each other as we please above and below a cut along the axis Ox. The function $z^{p-1}/(1+z)$ is single-valued within this boundary, which contains only one singular point, the pole $z = -1$. In order to calculate the value of the integral along this path, we shall agree to take for the angle of z that one which lies between 0 and 2π. If R denotes the residue with respect to the pole $z = -1$, we have then

$$\int_{ab}\frac{z^{p-1}}{1+z}\,dz + \int_{(C)}\frac{z^{p-1}\,dz}{1+z} + \int_{b'a'}\frac{z^{p-1}}{1+z}\,dz + \int_{(C')}\frac{z^{p-1}\,dz}{1+z} = 2i\pi R.$$

* Replace t by $1/(1+x)$ in the last formula of §135, Vol. I, wd ed.; §134, 1st ed. The formula (39), derived by supposing p to be real, is correct, provided the real part of p lies between 0 and 1.

The integrals along the circles C and C' approach zero as r becomes infinite and as ρ approaches zero respectively, for the product $z^p/(1+z)$ approaches zero in either case, since $0 < p < 1$.

[[101]] Along ab, z is real. For simplicity let us replace z by x. Since the angle of z is zero along ab, z^{p-1} is equal to the numerical value of x^{o-1}. Along $a'b'$ also z is real, but since its angle is 2π, we have [[101]]

$$z^{p-1} = e^{(p-1)(\log x + 2\pi i)} = e^{2\pi i(p-1)} x^{p-1}.$$

The sum of the two integrals along ab and along $a'b'$ therefore has for its limit

$$[1 - e^{2\pi i(p-1)}] \int_0^{+\infty} \frac{x^{p-1}}{1+x}\, dx.$$

The residue R is equal to $(-1)^{p-1}$, that is, to $e^{(p-1)\pi i}$, if π is taken as the angle of -1. We have, then,

$$\int_0^{+\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{2\pi i e^{(p-1)\pi i}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i}{e^{-(p-1)\pi i} - e^{(p-1)\pi i}} = \frac{-\pi}{\sin(p-1)\pi},$$

or, finally,

$$\int_0^{+\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}. \tag{39}$$

48. Application to functions analytic except for poles. Given two functions, $f(z)$ and $\phi(z)$, let us suppose that one of them, $f(z)$, is analytic except for poles in the interior of a closed curve C, that the other $\phi(z)$, is everywhere analytic within the same curve, and that the three functions $f(z)$, $f'(z)$, $\phi(z)$ are continuous on the curve C; and let us try to find the singular points of the function $\phi(z)f'(z)/f(z)$ within C. A point a which is neither a pole nor a zero for $f(z)$ is evidently an ordinary point for the function $f'(z)/f(z)$ and consequently for the function $\phi(z)f'(z)/f(z)$. If a point a is a pole or a zero of $f(z)$, we shall have, in the neighborhood of that point,

$$f(z) = (z-a)^\mu \psi(z),$$

where μ denotes a positive or negative integer equal to the order of the function at that point (§41), and where $\psi(z)$ is a regular function which is not zero for $z = a$. Taking the logarithmic derivatives on both sides, we find

$$\frac{f'(z)}{f(z)} = \frac{\mu}{z-a} + \frac{\psi'(z)}{\psi(z)}.$$

Since, on the other hand, we have, in the neighborhood of the point a,

$$\phi(z) = \phi(a) + (z-a)\phi'(a) + \cdots,$$

it follows that the point a is a pole of the first order for the product $\phi(z)f'(z)/f(z)$, and its residue is equal to $\mu\phi(a)$, that is, to $m\phi(a)$, if the point a is a zero of order m for $f(z)$, and to $-n\phi(a)$ if the point a is a
[[102]] pole of order n for $f(z)$. Hence, by the general theorem of residues, provided there are no roots of $f(z)$ on [[102]]
the curve C, we have

$$\frac{1}{2\pi i} \int_{(C)} \phi(z) \frac{f'(z)}{f(z)}\, dz = \sum \phi(a) - \sum \phi(b) \tag{40}$$

where a is any one of the zeros of $f(z)$ inside the boundary C, b any one of the poles of $f(z)$ within C, and where each of the poles and zeros are counted a number of times equal to its degree of multiplicity. The formula (40) furnishes an infinite number of relations, since we may take for $\phi(z)$ any analytic function.

Let us take in particular $\phi(z) = 1$; then the preceding formula becomes

$$N - P = \frac{1}{2\pi i} \int_{(C)} \frac{f'(z)}{f(z)}\, dz, \tag{41}$$

where N and P denote respectively the number of zeros and the number of poles of $f(z)$ within the boundary C. This formula leads to an important theorem. In fact, $f'(z)/f(z)$ is the derivative of $\operatorname{Log}[f(z)]$; to calculate the definite integral on the right-hand side of the formula (41) it is therefore sufficient to know the variation of

$$\log|f(z)| + i \operatorname{angle}[f(z)]$$

when the variable z describes the boundary C in the positive sense. But $|f(z)|$ returns to its initial value, while the angle of $f(z)$ increases by $2K\pi$, K being a positive or negative integer. We have, therefore,

$$N - P = \frac{2K\pi i}{2\pi i} = K; \tag{42}$$

that is, *the difference $N - P$ is equal to the quotient obtained by the division of the variation of the angle of $f(z)$ by 2π when the variable z describes the boundary C in the positive sense.*

Let us separate the real part and the coefficient of i in $f(z)$:

$$f(z) = X + Yi.$$

When the point $z = x + yi$ describes the curve C in the positive sense, the point whose coördinates are X, Y, with respect to a system of rectangular axes with the same orientation as the first system, describes also a closed curve C_1, and we need only draw the curve C_1 approximately in order to deduce from it by simple inspection the integer K. In fact, it is only necessary to count the number of revolutions which the radius vector joining the origin of coördinates to the point (X, Y) has turned through in one sense or the other.
[[103]] We can also write the formula (42) in the form [[103]]

$$N - P = \frac{1}{2\pi}\int_{(C)} d\arctan\left(\frac{Y}{X}\right) = \frac{1}{2\pi}\int_{(C)} \frac{X\,dY - Y\,dX}{X^2 + Y^2}. \tag{43}$$

Since the function Y/X takes on the same value after z has described the closed curve C, the definite integral

$$\int_{(C)} \frac{X\,dY - Y\,dX}{X^2 + Y^2}$$

is equal to $\pi I(Y/X)$, where the symbol $I(Y/X)$ means the index of the quotient Y/X along the boundary C, that is, the excess of the number of times that that quotient becomes infinite by passing from $+\infty$ to $-\infty$ over the number of times that it becomes infinite by passing from $-\infty$ to $+\infty$ (I, §§79, 154, 2d ed.; §§77, 154, 1st ed.). We can write the formula (43), then, in the equivalent form

$$N - P = \frac{1}{2} I\left(\frac{Y}{X}\right). \tag{44}$$

49. Application to the theory of equations. When the function $f(z)$ is itself analytic within the curve C, and has neither poles nor zeros on the curve, the preceding formulæ contain only the roots of the equation $f(z) = 0$ which lie within the region bounded by C. The formulæ (42), (43), and (44) show the number N of these roots by means of the variation of the angle of $f(z)$ along the curve or by means of the index Y/X.

If the function $f(z)$ is a polynomial in z, with any coefficients whatever, and when the boundary C is composed of a finite number of segments of unicursal curves, this index can be calculated by elementary operations, that is, by multiplications and divisions of polynomials. In fact, let AB be an arc of the boundary which can be represented by the expressions

$$x = \phi(t), \qquad y = \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are rational functions of a parameter t which varies from α to β as the point (x, y) describes the arc AB in the positive sense. Replacing z by $\phi(t) + i\psi(t)$ in the polynomial $f(z)$, we have

$$f(z) = R(t) + iR_1(t),$$

where $R(t)$ and $R_1(t)$ are rational functions of t with real coefficients. Hence the index of Y/X along the arc
[[104]] AB is equal to the index of the rational function R_1/R as t varies from α to β, which we already know how [[104]]
to calculate (I, §79, 2d ed.; §77, 1st ed.). If the boundary C is composed of segments of unicursal curves, we need only calculate the index for each of these segments and take half of their sum, in order to have the number of roots of the equation $f(z) = 0$ within the boundary C.

Note. D'Alembert's theorem is easily deduced from the preceding results. Let us prove first a lemma which we shall have occasion to use several times. Let $F(z)$, $\Phi(z)$ be two functions analytic in the interior of the closed curve C, continuous on the curve itself, and such that along the entire curve C we have $|\Phi(z)| < |F(z)|$; under these conditions *the two equations*

$$F(z) = 0, \qquad F(z) + \Phi(z) = 0$$

have the same number of roots in the interior of C. For we have

$$F(z) + \Phi(z) = F(z)\left[1 + \frac{\Phi(z)}{F(z)}\right].$$

As the point z describes the boundary C, the point $Z = 1 + \Phi(z)/F(z)$ describes a closed curve lying entirely within the circle of unit radius about the point $Z = 1$ as center, since $|Z - 1| < 1$ along the entire curve C. Hence the angle of that factor returns to its initial value after the variable z has described the boundary C, and the variation of the angle of $F(z) + \Phi(z)$ is equal to the variation of the angle of $F(z)$. Consequently the two equations have the same number of roots in the interior of C.

Now let $f(z)$ be a polynomial of degree m with any coefficients whatever, and let us set

$$F(z) = A_0 z^m, \qquad \Phi(z) = A_1 z^{m-1} + \cdots + A_m, \qquad f(z) = F(z) + \Phi(z).$$

Let us choose a positive number R so large that we have

$$\left|\frac{A_1}{A_0}\right| \frac{1}{R} + \left|\frac{A_2}{A_0}\right| \frac{1}{R^2} + \cdots + \left|\frac{A_m}{A_0}\right| \frac{1}{R^m} < 1.$$

Then along the entire circle C, described about the origin as center with a radius greater than R, it is clear that $|\Phi/F| < 1$. Hence the equatino $f(z) = 0$ has the same number of roots in the interior of the circle C as the equation $F(z) = 0$, that is, m.

50. Jensen's formula. Let $f(z)$ be an analytic function except for poles in the interior of the circle C with the radius r about the origin as center, and analytic and without zeros on C. Let $a_1, a_2, \cdots, a_n$ be the zeros, and $b_1, b_2, \cdots, b_m$ the poles, of $f(z)$ in the interior of this circle, each being counted according to its degree of multiplicity. We shall suppose, moreover, that
[[105]] the origin is neither a pole nor a zero for $f(z)$. Let us evaluate the definite integral [[105]]

$$\text{(45)} \qquad \mathrm{I} = \int_{(C)} \operatorname{Log}\,[f(z)]\,\frac{dz}{z},$$

taken along C in the positive sense, supposing that the variable z starts, for example, from the point $z = r$ on the real axis, and that a definite determination of the angle of $f(z)$ has been selected in advance. Integrating by parts, we have

$$\text{(46)} \qquad I = \{\operatorname{Log}(z)\operatorname{Log}\,[f(z)]\}_{(C)} - \int_{(C)} \operatorname{Log}(z)\frac{f'(z)}{f(z)}\,dz,$$

where the first part of the right-hand side denotes the increment of the product $\operatorname{Log}(z)\operatorname{Log}\,[f(z)]$ when the variable z describes the circle C. If we take zero for the initial value of the angle of z, that increment is equal to

$$(\log r + 2\pi i)\{\operatorname{Log}\,[f(r)] + 2\pi i(n - m)\} - \log r \operatorname{Log}\,[f(r)] = 2\pi i \operatorname{Log}\,[f(r)] + 2\pi i(n - m)\log r - 4(n - m)\pi^2.$$

In order to evaluate the new definite integral, let us consider the closed curve Γ, formed by the circumference C, by the circumference c described about the origin with the infinitesimal radius ρ, and by the two borders ab, $a'b'$ of a cut made along the real axis from the point $z = \rho$ to the point $z = r$ (Fig. 19). We shall suppose for definiteness that $f(z)$ has neither poles nor zeros on that portion of the axis of reals. If it has, we need only make a cut making an infinitesimal angle with the axis of reals. The function $\operatorname{Log} z$ is analytic in the interior of Γ, and according to the general formula (40) we have the relation

$$\int_{(ab)} \operatorname{Log}(z)\frac{f'(z)}{f(z)}\,dz + \int_{(C)} \operatorname{Log}(z)\frac{f'(z)}{f(z)}\,dz + \int_{(b'a')} \operatorname{Log}(z)\frac{f'(z)}{f(z)}\,dz + \int_{(C)} \operatorname{Log}(z)\frac{f'(z)}{f(z)}\,dz = 2\pi i \operatorname{Log}\left(\frac{a_1a_2\cdots a_n}{b_1b_2\cdots b_m}\right).$$

The integral along the circle c approaches zero with ρ, for the product $z\operatorname{Log} z$ is infinitesimal with ρ. On the other hand, if the angle of z is zero along ab, it is equal to 2π along $a'b'$, and the sum of the two corresponding integrals has for limit

$$-\int_0^r 2\pi i\frac{f'(z)}{f(z)}\,dz = -2\pi i\operatorname{Log}[f(r)] + 2\pi i \operatorname{Log}[f(0)].$$

The remaining portion is

$$\int_{(C)} \operatorname{Log}(z)\frac{f'(z)}{f(z)}\,dz = 2\pi i\operatorname{Log}\left(\frac{a_1a_2\cdots a_n}{b_1b_1\cdots b_m}\right) + 2\pi i\operatorname{Log}\left[\frac{f(r)}{f(0)}\right],$$

and the formula (46) becomes

$$I = 2\pi i(n-m)\log r + 2\pi i\operatorname{Log}[f(0)] - 2\pi i\operatorname{Log}\left(\frac{a_1a_2\cdots a_n}{b_1b_1\cdots b_m}\right) - 4(n-m)\pi^2.$$

In order to integrate along the circle C, we can put $z = re^{i\phi}$ and let ϕ vary from 0 to 2π. It follows that $dz/z = i\,d\phi$. Let
[[106]] $f(z) = Re^{i\Phi}$, where R and Φ are continuous functions of ϕ along C. Equating the coefficients of i in the preceding relation, we [[106]]
obtain Jensen's formula*

$$(47)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log R\,d\phi = \log|f(0)| + \log\left|r^{n-m}\frac{b_1b_2\cdots b_m}{a_1a_2\cdots a_n}\right|,$$

in which there appear only ordinary Napierian logarithms.

When the function $f(z)$ is analytic in the interior of C, it is clear that the product $b_1b_2\cdots b_n$ should be replaced by unity, and the formula becomes

$$(48)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log R\,d\phi = \log|f(0)| + \log\left|\frac{r^n}{a_1a_2\cdots a_n}\right|.$$

This relation is interesting in that it contains only the absolute values of the roots of $f(z)$ within the circle C, and the absolute value of $f(z)$ along that circle and for the center of the same circle.

51. Lagrange's formula. Lagrange's formula, which we have already established by Laplace's method (I, §195, 2d ed.; §189, 1st ed.), can be demonstrated also very easily by means of the general theorems of Cauchy. The process which we shall use is due to Hermite.

Let $f(z)$ be an analytic function in a certain region D containing the point a. The equation

$$(49)\qquad F(z) = z - a - \alpha f(z) = 0,$$

where α is a variable parameter, has the root $z = a$, for $\alpha = 0$.† Let us suppose that $\alpha \neq 0$, and let C be a circle with the center a and the radius r lying entirely in the regino D and such that we have along the entire circumference $|\alpha f(z)| < |z-a|$. By the lemma proved in §49 the equation $F(z) = 0$ has the same number of roots within the curve C as the equation $z - a = 0$, that is, a single root. Let ζ denote that root, and let $\Pi(z)$ be an analytic function in the circle C.

The function $\Pi(z)/F(z)$ has a single pole in the interior of C, at the point $z = \zeta$, and the corresponding residue is $\Pi(\zeta)/F'(\zeta)$. From the general theorem we have, then,

$$\frac{\Pi(\zeta)}{F'(\zeta)} = \frac{1}{2\pi i}\int_{(C)} \frac{\Pi(z)\,dz}{F(z)} = \frac{1}{2\pi i}\int_{(C)} \frac{\Pi(z)dz}{z-a-\alpha f(z)}.$$

* *Acta mathematica*, Vol. XXII.

† It is assumed that $f(a)$ is not zero, for otherwise $F(z)$ would vanish when $z = a$ for *any* value of α and the following developments would not yield any results of interest. – Trans.

In order to develop the integral on the right in powers of α, we shall proceed exactly as we did to derive the
[[107]] Taylor development, and we shall write [[107]]

$$\frac{1}{z-a-\alpha f(z)} = \frac{1}{z-a} + \frac{\alpha f(z)}{(z-a)^2} + \cdots + \frac{[\alpha f(z)]^n}{(z-a)^{n+1}} + \frac{1}{z-a-\alpha f(z)}\left[\frac{\alpha f(z)}{z-a}\right]^{n+1}.$$

Substituting this value in the integral, we find

$$\frac{\Pi(\zeta)}{F'(\zeta)} = J_0 + \alpha J_1 + \cdots + \alpha^n J_n + R_{n+1},$$

where

$$J_0 = \frac{1}{2\pi i}\int_{(C)} \frac{\Pi(z)\,dz}{z-a}, \qquad \cdots, \qquad J_n = \frac{1}{2\pi i}\int_{(C)} \frac{[f(z)]^n \Pi(z)\,dz}{(z-a)^{n+1}},$$

$$R_{n+1} = \frac{1}{2\pi i}\int_{(C)} \frac{\Pi(z)}{z-a-\alpha f(z)}\left[\frac{\alpha f(z)}{z-a}\right]^{n+1} dz.$$

Let m be the maximum value of the absolute value of $\alpha f(z)$ along the circumference of the circle C; then, by hypothesis, m is less than r. If M is the maximum value of the absolute value of $\Pi(z)$ along C, we have

$$|R_{n+1}| < \frac{1}{2\pi}\left(\frac{m}{r}\right)^{n+1} \frac{2\pi r M}{r-m},$$

which shows that R_{n+1} approaches zero when n increases indefinitely. Moreover, we have, by the definition of the coefficients J_0, J_1, $\cdots$, J_n, $\cdots$ and the formula (14),

$$J_0 = \Pi(a), \qquad \cdots, \qquad J_n = \frac{1}{n!}\frac{d^n}{da^n}\left\{[f(a)]^n \Pi(a)\right\};$$

whence we obtain the following development in series:

$$\frac{\Pi(\zeta)}{F'(\zeta)} = \Pi(a) + \sum_{n=1}^{+\infty} \frac{\alpha^n}{n!}\frac{d^n}{da^n}\left\{\Pi(a)[f(a)]^n\right\}. \tag{50}$$

We can write this experssion in a somewhat different form. If we take $\Pi(z) = \Phi(z)[1-\alpha f'(z)]$, where $\Phi(z)$ is an analytic function in the same region, the left-hand side of the equation (50) will no longer contain α and will reduce to $\Phi(\zeta)$. As for the right-hand side, we observe that it contains two terms of degree n in α, whose sum is

$$\begin{aligned}
&\frac{\alpha^n}{n!}\frac{d^n}{da^n}\left\{\Phi(a)[f(a)]^n\right\} - \frac{\alpha^n}{(n-1)!}\frac{d^{n-1}}{da^{n-1}}\left\{\Phi(a)[f(a)]^{n-1} f'(a)\right\} \\
&\qquad = \frac{\alpha^n}{n!}\frac{d^{n-1}}{da^{n-1}}\left\{\Phi'(a)[f(a)]^n + n\Phi(a)f'(a)[f(a)]^{n-1} - n\Phi(a)f'(a)[f(a)]^{n-1}\right\} \\
&\qquad = \frac{\alpha^n}{n!}\frac{d^{n-1}}{da^{n-1}}\left\{\Phi'(a)[f(a)]^n\right\},
\end{aligned}$$

[[108]] and we find again Lagrange's formula in its usual form (see I, formula (52), §195, 2d ed.; §189, 1st ed.) [[108]]

$$\Phi(\zeta) = \Phi(a) + \frac{\alpha}{1}\Phi'(a)f(a) + \cdots + \frac{\alpha^n}{n!}\frac{d^{n-1}}{da^{n-1}}\left\{\Phi'(a)[f(a)]^n\right\} + \cdots. \tag{51}$$

We have supposed that we have $|\alpha f(z)| < r$ along the circle C, which is true if $|\alpha|$ is small enough. In order to find the maximum value of $|\alpha|$, let us limit ourselves to the case where $f(z)$ is a polynomial or an intgral function. Let $\mathcal{M}(r)$ be the maximum value of $|f(z)|$ along the circle C described about the point

a as center with the radius r. The proof will apply to this circle, provided $|\alpha|\mathcal{M}(r) < r$. We are thus led to seek the maximum value of the quotient $r/\mathcal{M}(r)$, as r varies from 0 to $+\infty$. This quotient is zero for $r = 0$, for if $\mathcal{M}(r)$ were to approach zero with r, the point $z = a$ would be a zero for $f(z)$, and $F(z)$ would vanish for $z = a$. The same quotient is also zero for $r = \infty$, for otherwise $f(z)$ would be a polynomial of the first degree (§36). Aside from these trivial cases, it follows that $r/\mathcal{M}(r)$ passes through a maximum value μ for a value of r_1 of r. The reasoning shows that the equation (49) has one and only one root ζ such that $|\zeta - a| < r_1$, provided $|\alpha| < \mu$. hence the developments (50) and (51) Are applicable so long as $|\alpha|$ does not exceed μ, provided the functions $\Pi(z)$ and $\Phi(z)$ are themselves analytic in the circle C_1 of radius r_1.

Example. Let $f(z) = (z^2 - 1)/2$; the equation (49) has the root

$$\zeta = \frac{1 - \sqrt{1 - 2a\alpha + \alpha^2}}{\alpha},$$

which approaches a when α approaches zero. Let us put $\Pi(z) = 1$. Then the formula (50) takes the form

$$\frac{1}{\sqrt{1 - 2a\alpha + \alpha^2}} = 1 + \sum_1^{+\infty} \frac{\alpha^n}{n!} \frac{d^n}{da^n}\left[\frac{(a^2-1)^n}{2^n}\right] = 1 + \sum_1^{+\infty} \alpha^n X_n(a), \tag{52}$$

where X_n is the nth Legendre's polynomial (see I, §§90, 189, 2d ed.; §§88, 184, 1st ed.). In order to find out between what limits the formula is valid, let us suppose that a is real and greater than unity. On the circle of radius r we have evidently $\mathcal{M}(r) = [(a + r)^2 - 1]/2$, and we are led to seek the maximum value of $2r/[(a + r)^2 - 1]$ as r increases from 0 to $+\infty$. This maximum is found for $r = \sqrt{a^2 - 1}$, and it is equal to $a - \sqrt{a^2 - 1}$. If, however, a lies between -1 and $+1$, we find by a quite elementary calculation that

$$\mathcal{M}(r) = \frac{r^2 + 1 - a^2}{2\sqrt{1 - a^2}}.$$

The maximum of $2r\sqrt{1 - a^2}/(r^2 + 1 - a^2)$ occurs when $r = \sqrt{1 - a^2}$, and it is equal to unity.

[[109]] It is easy to verify these results. In fact, the radical $\sqrt{1 - 2a\alpha + \alpha^2}$, considered as a function of α, has the two critical points $a \pm \sqrt{a^2 - 1}$. If $a > 1$, the critical point nearest the origin is $a - \sqrt{a^2 - 1}$. When a lies between -1 and $+1$, the absolute value of each of the two critical points $a \pm i\sqrt{1 - a^2}$ is unity. [[109]]

In the fourth lithographed section of Hermite's lectures will be found (p. 185) a very complete discussion of Kepler's equation $z - a = \sin z$ by this method. His process leads to the calculation of the roots of the transcendental equation $e^r(r - 1) = e^{-r}(r + 1)$ which lies between 1 and 2. Stieltjes has obtained the values

$$r_1 = 1.199678640257734, \qquad \mu = 0.6627434193492.$$

[[123]] [[123]]

EXERCISES

1. Develop the function

$$y = \frac{1}{2}\left(x + \sqrt{x^2 - 1}\right)^m + \frac{1}{2}\left(x - \sqrt{x^2 - 1}\right)^m$$

in powers of x, m being any number.

Find the radius of the circle of convergence.

2. Find the different developments of the function $1/[(z^2 + 1)(z - 2)]$ in positive or negative powers of z, according to the position of the point z in the plane.

3. Calculate the definite integral $\int z^2 \text{Log}\,[(z + 1)/(z - 1)]\,dz$ taken along a circle of radius 2 about the origin as center, the initial value of the logarithm at the point $z = 2$ being taken as real.

Calculate the definite integral

$$\int \frac{dz}{\sqrt{z^2 + z + 1}}$$

taken over the same boundary.

4. Let $f(z)$ be an analytic function in the interior of a closed curve C containing the origin. Calculate the definite integral $\int_{(C)} f'(z)\text{Log}\,z\,dz$, taken along the curve C, starting with an initial value z_0.

5. Derive the relation

$$\int_{-\infty}^{+\infty} \frac{dt}{(1 + t^2)^{n+1}} = \frac{1.3.5\cdots(2n - 1)}{2.4.6\cdots 2n}\pi$$

and deduce from it the definite integrals

$$\int_{-\infty}^{+\infty} \frac{dt}{[(t-\alpha)^2+\beta^2]^{n+1}}, \quad \int_{-\infty}^{+\infty} \frac{dt}{(At^2+2Bt+C)^{n+1}}.$$

6. Calculate the following definite integrals by means of the theory of residues:

$$\int_0^{+\infty} \frac{\sin mx\,dx}{x(x^2+a^2)^2}, \qquad m \text{ and } a \text{ being real,}$$

$$\int_{-\infty}^{+\infty} \frac{\cos ax}{1+x^4}\,dx, \qquad a \text{ being real,}$$

[[124]]

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2-2\beta ix-\beta^2-\alpha^2)^{n+1}}, \qquad \alpha \text{ and } \beta \text{ being real,}$$

$$\int_{-\infty}^{+\infty} \frac{\cos x\,dx}{(x^2+1)(x^2+4)},$$

$$\int_0^1 \frac{\sqrt[3]{4x^2(1-x)}}{(1+x)^3}\,dx, \qquad \int_0^{+\infty} \frac{x\log x\,dx}{(1+x^2)^3},$$

$$\int_0^{+\infty} \frac{\cos ax-\cos bx}{x^2}\,dx, \qquad a \text{ and } b \text{ being real and positive.}$$

(To evaluate the last integral, integrate the function $(e^{aiz}-e^{biz})/z^2$ along the boundary indicated by Fig. 17.)

7. The definite integral $\int_0^\pi d\phi/[A+C-(A-C)\cos\phi]$ is equal, when it has any finite value, to $\epsilon\pi/\sqrt{AC}$, where ϵ is equal to ±1 and is chosen in such a way that the coefficient of i in $\epsilon i\sqrt{AC}/A$ is positive.

8. Let $F(z)$ and $G(z)$ be two analytic functions, and $z=a$ a double root of $G(z)=0$ that is not a root of $F(z)$. Show that the corresponding residue of $F(z)/G(z)$ is equal to

$$\frac{6F'(a)G''(a)-2F(a)G'''(a)}{3[G''(a)]^2}.$$

In a similar manner show that the residue of $F(z)/[G(z)]^2$ for a simple root a of $G(z)=0$ is equal to

$$\frac{F'(a)G'(a)-F(a)G''(a)}{[G'(a)]^3}.$$

9. Derive the formula

$$\int_{-1}^{+1} \frac{dx}{(x-a)\sqrt{1-x^2}} = \frac{\pi i}{\sqrt{1-a^2}},$$

the integral being taken along the real axis with the positive value of the radical, and a being a complex number or a real number whose absolute value is greater than unity. Determine the value that should be taken for $\sqrt{1-a^2}$.

10. Consider the integrals $\int_{(S)} dz/\sqrt{1+z^3}$, $\int_{(S_1)} dz/\sqrt{1+z^3}$, where S and S_1 denote two boundaries formed as follows: The boundary S is composed of a straight-line segment OA on Ox (which is made to expand indefinitely), of the circle of radius OA about O as center, and finally of the straight line AO. The boundary S_1 is the succession of three loops which inclose the points a, b, c which represent the roots of the equation $z^3+1=0$.

Establish the relation that exists between the two integrals

$$\int_0^{+\infty} \frac{dx}{\sqrt{1+x^3}}, \qquad \int_0^1 \frac{dt}{\sqrt{1-t^3}},$$

which arise in the course of the preceding consideration.

11. By integrating the function e^{-z^2} along the boundary of the rectangle formed by the straight lines $y=0$, $y=b$, $x=+R$, $x=-R$, and then making R become infinite, establish the relation

$$\int_{-\infty}^{+\infty} e^{-x^2}\cos 2bx\,dx = \sqrt{\pi}e^{-b^2}.$$

[[125]] **12**. Integrate the function $e^{-z}z^{n-1}$, where n is real and positive, along a boundary formed by a radius OA placed along Ox, by an arc of a circle AB of radius OA about O as center, and by a radius BO such that the angle $\alpha = AOB$ lies between 0 and $\pi/2$. Making OA become infinite, deduce from the preceding the values of the definite integrals [[125]]

$$\int_0^{+\infty} u^{n-1}e^{-au}\cos bu\,du, \qquad \int_0^{+\infty} u^{n-1}e^{-au}\sin bu\,du,$$

where a and b are real and positive. The results obtained are valid for $\alpha = \pi/2$, provided that we have $n < 1$.

13. Let m, m', n be positive integers ($m < n$, $m' < n$). Establish the formula

$$\int_0^{+\infty} \frac{t^{2m} - t^{2m'}}{1 - t^{2n}} = \frac{\pi}{2n}\left[\operatorname{ctn}\left(\frac{2m+1}{2n}\pi\right) - \operatorname{ctn}\left(\frac{2m'+1}{2n}\pi\right)\right].$$

14. Deduce from the previous result Euler's formula

$$\int_0^{+\infty} \frac{t^{2m}\,dt}{1 + t^{2n}} = \frac{\pi}{2n\sin\left(\frac{2m+1}{2n}\pi\right)}.$$

15. If the real part of a is positive and less than unity, we have

$$\int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{1 + e^x} = \frac{\pi}{\sin a\pi}.$$

(This can be deduced from the formula (39) (§47) or by integrating the function $e^{az}/(1 + e^z)$ along the boundary of the rectangle formed by the straight lines $y = 0$, $y = 2\pi$, $x = +R$, $x = -R$, and then making R become infinite.)

16. Derive in the same way the relation

$$\int_{-\infty}^{+\infty} \frac{e^{ax} - e^{bx}}{1 - e^x}\,dx = \pi(\operatorname{ctn} a\pi - \operatorname{ctn} b\pi),$$

where the real parts of a and b are positive and less than unity.

(Take for the path of integration the rectangle formed by the straight lines $y = 0$, $y = \pi$, $x = R$, $x = -R$, and make use of the preceding exercise.)

17. From the formula

$$\int_{(C)} \frac{(1+z)^n}{z^{k+1}}\,dz = 2\pi i\frac{n(n-1)\cdots(n-k+1)}{k!},$$

where n and k are positive integers, and C is a circle having the origin as center, deduce the relations

$$\int_0^{\pi} (2\cos u)^{n+k}\cos(n-k)u\,du = \pi\frac{(n+1)(n+2)\cdots(n+k)}{k!},$$

$$\int_{-1}^{+1} \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \pi\frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}.$$

(Put $z = e^{2iu}$, then $\cos u = x$, and replace n by $n + k$, and k by n.)

18*. The definite integral

$$\Phi(z) = \int_0^{\pi} \frac{d\phi}{1 - \alpha\left(x + \sqrt{x^2-1}\cos\phi\right)},$$

[[126]] when it has a finite value, is equal to $\pm\pi/\sqrt{1 - 2\alpha x + \alpha^2}$, where the sign depends upon the relative positions of the two points of α and x. Deduce from this the expression, due to Jacobi, for the nth Legendre's polynomial, [[126]]

$$X_n = \frac{1}{\pi}\int_0^{\pi} \left(x + \sqrt{x^2-1}\cos\phi\right)^n d\phi.$$

19. Study in the same way the definite integral

$$\int_0^\pi \frac{d\phi}{x - a + \sqrt{x^2-1}\cos\phi},$$

and deduce from the result Laplace's formula

$$X_n = \frac{\epsilon}{\pi}\int_0^\pi \frac{d\phi}{\left(x + \sqrt{x^2-1}\cos\phi\right)^{n+1}},$$

where $\epsilon = \pm 1$, according as the real part of x is positive or negative.

20*. Establish the last result by integrating the function

$$\frac{1}{z^{n+1}\sqrt{1-2xz+z^2}}$$

along a circle about the origin as center, whose radius is made to become infinite.

21*. **Gauss's sums.** Let $T_8 = e^{2\pi i s^2/n}$, where n and s are integers; and let S_n denote the sum $T_0 + T_1 + \cdots + T_{n-1}$. Derive the formula

$$S_n = \frac{(1+i)(1+i^{3n})}{2}\sqrt{n}.$$

(Apply the theorem on residues to the function $\Phi(z) = e^{2\pi i z^2/n}/(e^{2\pi i z} - 1)$, taking for the boundary of integration the sides of the rectangle formed by the straight lines $x = 0$, $x = n$, $y = +R$, $y = -R$, and inserting two semicircumferences of radius ϵ about the points $x = 0$, $x = n$ as centers, in order to avoid the poles $z = 0$ and $z = n$ of the function $\phi(z)$; then let R become infinite.)

22. Let $f(z)$ be an analytic function in the interior of a closed curve Γ containing the points a, b, c, $\cdots$, l. If α, β, $\cdots$, λ are positive integers, show that the sum of the residues of the function

$$\phi(z) = \frac{f(z)}{x-z}\left(\frac{x-a}{z-a}\right)^\alpha\left(\frac{x-b}{z-b}\right)^\beta\cdots\left(\frac{x-l}{z-l}\right)^\lambda$$

with respect to the poles a, b, c, $\cdots$, l is a polynomial $F(x)$ of degree

$$\alpha + \beta + \cdots + \lambda - 1$$

satisfying the relations

$$\begin{array}{llll} F(a) = f(a), & F'(a) = f'(a), & \cdots, & F^{(\alpha-1)}(a) = f^{(\alpha-1)}(a), \\ F(b) = f(b), & F'(b) = f'(b), & \cdots, & F^{(\beta-1)}(b) = f^{(\beta-1)}(b), \\ \cdots, & \cdots, & \cdots, & \cdots. \end{array}$$

(Make use of the relation $F(x) = f(x) + [\int_{(\Gamma)} \phi(z)\,dz]/2\pi i$.)