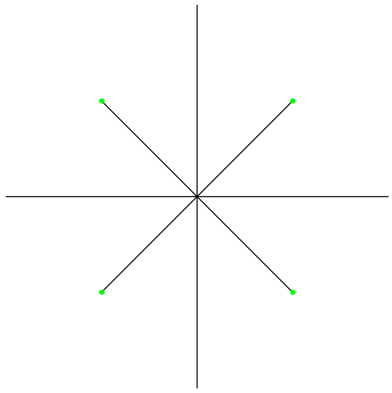
1. (a) [16 marks] Find all complex numbers z which satisfy the equation $1 + z^4 = 0$, and plot them on the complex plane.

We are given $z^4 + 1 = 0$, so $z^4 = -1$. Suppose that $z = r(\cos\theta + i\sin\theta)$; then $z^4 = r^4(\cos 4\theta + i\sin 4\theta)$, so we must have $r^4(\cos 4\theta + i\sin 4\theta) = \cos\pi + i\sin\pi$, hence $r = 1$, $4\theta = \pi + 2n\pi$ for some $n \in \mathbf{Z}$, or $\theta = \pi/4 + n\pi/2$, $n \in \mathbf{Z}$. Since values of θ differing by 2π represent the same point z, we have four distinct values for z, with $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$. These can be drawn as follows:



[2 marks for each correct expression for a root, 2 marks for each correct plotted root.]

(b) [16 marks] Write $z = x + iy$, expand out z^4, and use the Cauchy-Riemann equations to show that $1 + z^4$ is analytic at all points in the complex plane. [This part corresponds to Quiz 1.]

We have

$$(x+iy)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k}(iy)^k = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$
$$= x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3). \qquad \text{[4 marks]}$$

Letting P denote the real and Q the imaginary part of $1 + z^4$, we have [2 marks/derivative, 8 marks total]

$$\frac{\partial P}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial P}{\partial y} = -12x^2y + 4y^3, \quad \frac{\partial Q}{\partial x} = 12x^2y - 4y^3, \quad \frac{\partial Q}{\partial y} = 4x^3 12xy^2,$$

so [1 mark/equation, 2 marks total]

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

and since the derivatives are continuous everywhere [2 marks], we see that $1 + z^4$ is analytic on the whole complex plane.

[Marking: as indicated.]

(c) [4 marks] Use the results of (a) and (b) to find the region in the complex plane where the function

$$\frac{1}{1+z^4}$$

is analytic (in the sense of having a complex derivative).

By the quotient rule and (b)[1 mark], $1/(1 + z^4)$ will be analytic as long as the denominator is nonzero [1 mark]. By (a), this will happen when $z \neq \cos(\pi/4 + n\pi/2) + i\sin(\pi/4 + n\pi/2)$ for any $n \in \mathbf{Z}$ [1 mark]. Thus $1/(1 + z^4)$ is analytic on $\mathbf{C}\backslash\{e^{i(\pi/4+n\pi/2)} \,|\, n \in \mathbf{Z}\}$.[1 mark]

[Marking: as indicated.]

The following is for Question 2 and Question 3:

Now let us define a (potentially multi-valued) function f of the complex variable z by the rule

$$f(z) = \int_0^z \frac{1}{1+z'^4}\, dz',$$

where the value of f may depend on the curve chosen from 0 to z (if so, then f will be multi-valued). If x is a real number, let

$$g(x) = \int_0^x \frac{1}{1+u^4}\, du,$$

where the integral is the usual real-variable integral; in other words, $g(x)$ is $f(x)$ where the contour defining f is required to lie along the real axis.

2. (a) [12 marks] Consider the function $f(iy)$, where the contour is taken along the imaginary axis. By parameterising this contour, show how to express $f(iy)$ in terms of $g(y)$.

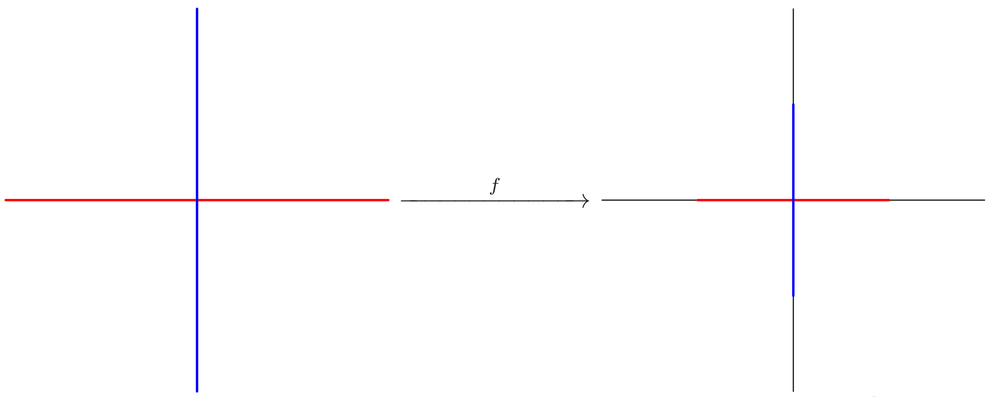
Let $y \in \mathbf{R}$, and define the curve $\gamma : [0,1] \to \mathbf{C}$ by $\gamma(t) = ity$ [2 marks]; then γ will be a curve along the imaginary axis from 0 to iy, and along this curve we have, since $\gamma'(t) = iy$[1 mark],

$$\int_\gamma \frac{1}{1+z^4}\, dz = \int_0^1 \frac{1}{1+(ity)^4} iy\, dt\text{[2 marks]} = \int_0^1 \frac{1}{1+(ty)^4} iy\, dt\text{[2 marks]}$$
$$= i\int_0^y \frac{1}{1+u^4}\, du\text{[3 marks]} = ig(y)\text{[2 marks]}.$$

[Marking: as indicated.]

(b) [12 marks] Use your result from (a) to draw the image of the real and imaginary axes under the function $z \mapsto f(z)$, where in both cases we require the contours used to lie along the respective axes. (You may assume that the function g maps the real line onto some open interval around 0.)

Evidently, along the real axis, $f(x) = g(x)$ [2 marks], while from (a), $f(iy) = ig(y)$ [2 marks]. Thus f will take the real axis into some open interval around 0 on the real axis [2 marks], and the imaginary axis into some open interval around 0 on the imaginary axis [2 marks]. If we indicate this pictorially, we have



[2 marks each for correctly indicating the mapping of the real and imaginary axis]

[Marking: as indicated. Some explanation in words of the image sets should be given, but it can be less formal than what is here.]

(c) [8 marks] What is $f'(0)$? Can we conclude that f is conformal at $z = 0$? Explain how this relates to your picture in (b).

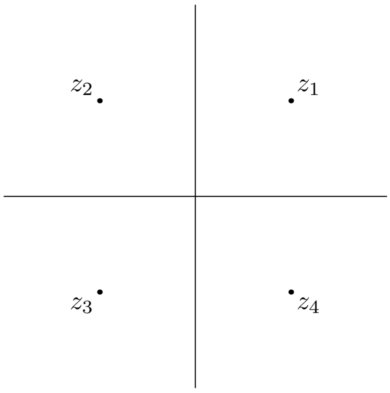
Since $1/(1+z^4)$ is analytic near $z = 0$ [2 marks], we will have $f'(z) = 1/(1+z^4)$ for z near 0, and in particular $f'(0) = 1/(1+0^4) = 1/1 = 1$ [2 marks]. Since f' exists and is nonzero, we see that f is conformal at $z = 0$ [2 marks]. This is exemplified by the fact that the image 'curves' (in this case, line segments) in (b) make the same angle with each other as the original ones do [2 marks].

[Marking: as indicated.]

3. This is a continuation of Question 2.

(a) [8 marks] Find all points z at which f does not possess a complex derivative. (You should give a reason for your answer, but you do not need to give a full proof.) Plot these points on the complex plane. Label them z_1, z_2, z_3, z_4, in any order you wish. [Hint: it will be useful to have z_1 and z_2 lie on the same side of the real axis.]

By the fundamental theorem of calculus [2 marks] [Goursat, §31], f will have a derivative at every point at which $1/(1+z^4)$ does [2 marks]; hence it must have a derivative at every point except the points $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$, as we discussed in 1(c) above [2 marks]. We plot these as follows: [2 marks]



[Marking: as indicated. A citation to Goursat or anywhere else is of course not required.]

(b) [16 marks] Use your solution to 1(a) to factor $1 + z^4$, and then apply the Cauchy integral formula to determine the value of the integral

$$\int_\gamma \frac{1}{1 + z'^4}\, dz',$$

where γ is any curve enclosing z_1 but none of the other points you found in (a). Simplify your answer as much as possible.

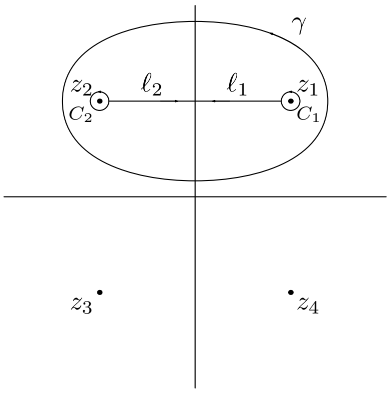
We see that we may write $1 + z^4 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$ [4 marks]. Since γ does not enclose z_2, z_3, or z_4, the function $1/(z - z_2)(z - z_3)(z - z_4)$ will be analytic on and within γ, and hence we may apply the Cauchy integral formula to write[2 marks]

$$\begin{aligned}
\int_\gamma \frac{1}{1 + z^4}\, dz &= \int_\gamma \frac{\frac{1}{(z-z_2)(z-z_3)(z-z_4)}}{z - z_1}\, dz \qquad \text{[3 marks]} \\
&= \frac{2\pi i}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \qquad \text{[2 marks]} \\
&= \frac{2\pi i}{\sqrt{2}(\sqrt{2} + i\sqrt{2})(\sqrt{2}i)} = \frac{2\pi i}{2^{3/2}(-1 + i)} \\
&= \pi \frac{1 - i}{2^{3/2}}. \qquad \text{[5 marks]}
\end{aligned}$$

[Marking: as indicated. The answer must be simplified to a numeric form for full marks.]

(c) [32 marks] Repeat (b), but now let γ enclose only z_1 and z_2. [Hint: can you see how to use the Cauchy integral theorem to replace γ with two small circles around z_1 and z_2?]

Consider the following picture: [2 marks for each of the curves C_1, C_2]



By the Cauchy integral theorem, we have

$$\int_\gamma \frac{1}{1 + z^4}\, dz = \int_{C_1} \frac{1}{z + z^4}\, dz + \int_{C_2} \frac{1}{1 + z^4}\, dz + \int_{\ell_1} \frac{1}{1 + z^4}\, dz + \int_{\ell_2} \frac{1}{1 + z^4}\, dz;$$

now the last two integrals cancel, since $1/(1+z^4)$ is continuous along the line ℓ_1 and ℓ_2, and so we have

$$\int_\gamma \frac{1}{1+z^4}\,dz = \int_{C_1} \frac{1}{1+z^4}\,dz + \int_{C_2} \frac{1}{1+z^4}\,dz. \qquad \text{[4 marks]}$$

But now $\int_{C_1} \frac{1}{1+z^4}\,dz = \pi\frac{1-i}{2^{3/2}}$[10 marks] by (b); and $\int_{C_2} \frac{1}{1+z^4}\,dz$ can be computed in the same way:

$$\begin{aligned}\int_{C_2} \frac{1}{1+z^4}\,dz &= \int_{C_2} \frac{\frac{1}{(z-z_1)(z-z_3)(z-z_4)}}{z-z_2}\,dz \qquad \text{[3 marks]}\\ &= \frac{2\pi i}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\text{[2 marks]} = \frac{2\pi i}{-\sqrt{2}(\sqrt{2}i)(-\sqrt{2}+\sqrt{2}i)}\\ &= -\frac{2\pi}{2^{3/2}(-1+i)} = \pi\frac{1+i}{2^{3/2}}, \qquad \text{[5 marks]}\end{aligned}$$

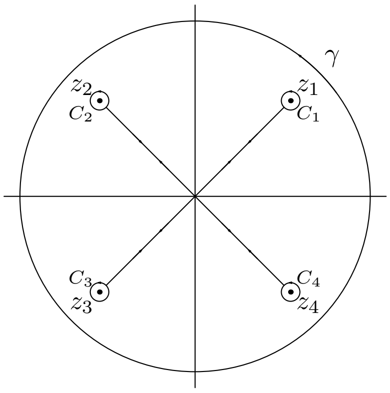
so

$$\int_\gamma \frac{1}{1+z^4}\,dz = -\pi i\frac{(1+i)^2}{2^{3/2}} = \frac{\pi}{2^{1/2}}. \qquad \text{[4 marks]}$$

[Marking: as indicated. The lines ℓ_1 and ℓ_2 do not need to be used for full marks.]

(d) [32 marks] Repeat (b), but now let γ enclose all four points.

Consider the following picture: [1 mark for each curve C_1]



By the same logic as in (c), we see that

$$\int_\gamma \frac{1}{1+z^4}\,dz = \int_{C_1} \frac{1}{1+z^4}\,dz + \int_{C_2} \frac{1}{1+z^4}\,dz + \int_{C_3} \frac{1}{1+z^4}\,dz + \int_{C_4} \frac{1}{1+z^4}\,dz. \qquad \text{[4 marks]}$$

We have already calculated the first two integrals; the remaining two may be calculated similarly:

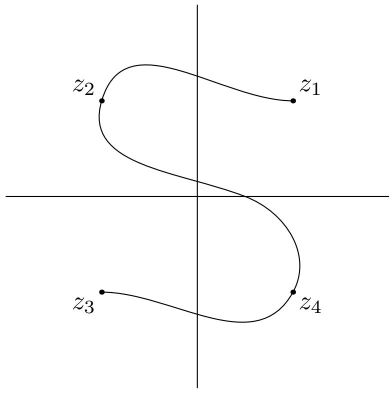
$$\begin{aligned}\int_{C_3} \frac{1}{1+z^4}\,dz &= \int_{C_3} \frac{\frac{1}{(z-z_1)(z-z_2)(z-z_4)}}{z-z_3}\,dz \qquad \text{[3 marks]}\\ &= \frac{2\pi i}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}\text{[2 marks]} = \frac{2\pi i}{-(\sqrt{2}+\sqrt{2}i)(-\sqrt{2}i)(-\sqrt{2})}\\ &= \frac{2\pi i}{2^{3/2}(1-i)} = \pi\frac{-1+i}{2^{3/2}} \qquad \text{[5 marks]}\\ \int_{C_4} \frac{1}{1+z^4}\,dz &= \frac{2\pi i}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}\text{[5 marks]} = \frac{2\pi i}{(-\sqrt{2}i)(\sqrt{2}-\sqrt{2}i)(\sqrt{2})}\\ &= -\frac{2\pi i}{2^{3/2}(1+i)} = -\pi\frac{1+i}{2^{3/2}}, \qquad \text{[5 marks]}\end{aligned}$$

and we see that $\int_\gamma \frac{1}{1+z^4}\,dz = 0$.[4 marks]

[Marking: as indicated. Again, the lines between the curves do not need to be given for full marks.]

(e) [24 marks] Now let C be any simple (non-selfintersecting) piecewise-smooth curve whose endpoints are any two of the points z_1, z_2, z_3, z_4, which also passes through the remaining two points [Hint: it will be useful to have it start at z_1 and go to z_2 next], and which crosses the real axis exactly once, at some point $x_0 > 0$. Draw this curve on the plane you drew in (a). Let $D = \mathbf{C}\backslash C$ denote the complex plane with the curve C removed. Use your result from (d), together with the Cauchy integral theorem if necessary, to show that if we require the contour in the definition of f to be strictly within D, then f becomes a single-valued function.

We choose the following curve: [2 marks for each of the following: (i) initial point is one of the z_i; (ii) curve passes through all four z_i; (iii) end point is one of the z_i; (iv) curve only crosses the real axis once.]



Now let z be any point in the plane not lying on C, and let γ_1, γ_2 be two curves in $\mathbf{C}\backslash C$ from 0 to z [2 marks]. Then if any of the points z_1, z_2, z_3, z_4 lies in between γ_1 and γ_2, the whole curve C must also lie between them [6 marks]; by (d), then, the integral

$$\int_{\gamma_1-\gamma_2} \frac{1}{1+z^4}\,dz = \int_{\gamma_1} \frac{1}{1+z^4}\,dz - \int_{\gamma_2} \frac{1}{1+z^4}\,dz$$

must vanish[3 marks]. If none of the points lie between γ_1 and γ_2, then this integral will vanish by the Cauchy integral theorem[3 marks]. Thus in either case,

$$\int_{\gamma_1} \frac{1}{1+z^4}\,dz = \int_{\gamma_2} \frac{1}{1+z^4}\,dz,$$

and f will be single-valued.[2 marks]

[Marking: as indicated.]

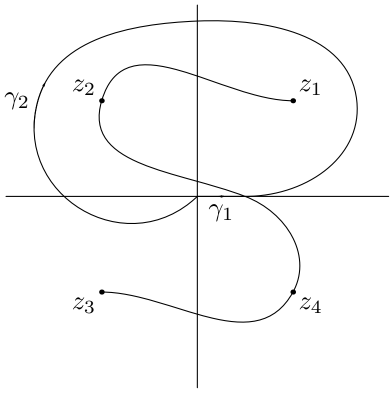
(f) [16 marks] Using the single-valued version of f described in (e), calculate

$$\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z),$$

where $z \to x_0^\pm$ means that z approaches from the right (+) or left (−) of the curve C. [Hint: can you see how to apply your result from (c)?] Does this difference of limits depend on your choice of C? (You do not need to give a justification.)

[2(a) and 3(f) correspond to Quiz 2.]

Since f is single-valued, we may use any two curves γ_1, γ_2[3 marks]. We use the following curves: [2 marks for each curve; curves should form a closed loop containing only two of the z_i – it doesn't matter which two – and not intersecting the curve C except at its intersection with the real axis]



Now $1/(1+z^4)$ is continuous at x_0, so in the limit

$$\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z) = \int_{\gamma_2} \frac{1}{1+z^4}\,dz - \int_{\gamma_1} \frac{1}{1+z^4}\,dz$$

$$\text{[2 marks]} = -\int_{\gamma_1-\gamma_2} \frac{1}{1+z^4}\,dz \text{[3 marks]} = -\frac{\pi}{2^{1/2}} \text{[2 marks]}$$

by (c). It is quite clear that this does not depend on the choice of C, since regardless of the choice of C the curve $\gamma_1-\gamma_2$ must enclose exactly the two points z_1 and z_2, which entirely determines the integral of $1/(1+z^4)$ over $\gamma_1-\gamma_2$.[2 marks]

[Marking: as indicated.]