1. (a) [16 marks] Find all complex numbers z which satisfy the equation $1 + z^4 = 0$, and plot them on the complex plane.

(b) [16 marks] Write $z = x + iy$, expand out z^4, and use the Cauchy-Riemann equations to show that $1 + z^4$ is analytic at all points in the complex plane. [This part corresponds to Quiz 1.]

(c) [4 marks] Use the results of (a) and (b) to find the region in the complex plane where the function

$$\frac{1}{1+z^4}$$

is analytic (in the sense of having a complex derivative).

The following is for Question 2 and Question 3:

Now let us define a (potentially multi-valued) function f of the complex variable z by the rule

$$f(z) = \int_0^z \frac{1}{1+z'^4}\, dz',$$

where the value of f may depend on the curve chosen from 0 to z (if so, then f will be multi-valued). If x is a real number, let

$$g(x) = \int_0^x \frac{1}{1+u^4}\, du,$$

where the integral is the usual real-variable integral; in other words, $g(x)$ is $f(x)$ where the contour defining f is required to lie along the real axis.

2. (a) [12 marks] Consider the function $f(iy)$, where the contour is taken along the imaginary axis. By parameterising this contour, show how to express $f(iy)$ in terms of $g(y)$.

(b) [12 marks] Use your result from (a) to draw the image of the real and imaginary axes under the function $z \mapsto f(z)$, where in both cases we require the contours used to lie along the respective axes. (You may assume that the function g maps the real line onto some open interval around 0.)

(c) [8 marks] What is $f'(0)$? Can we conclude that f is conformal at $z = 0$? Explain how this relates to your picture in (b).

3. This is a continuation of Question 2.

(a) [8 marks] Find all points z at which f does not possess a complex derivative. (You should give a reason for your answer, but you do not need to give a full proof.) Plot these points on the complex plane. Label them z_1, z_2, z_3, z_4, in any order you wish. [Hint: it will be useful to have z_1 and z_2 lie on the same side of the real axis.]

(b) [16 marks] Use your solution to 1(a) to factor $1 + z^4$, and then apply the Cauchy integral formula to determine the value of the integral

$$\int_\gamma \frac{1}{1+z'^4}\, dz',$$

where γ is any curve enclosing z_1 but none of the other points you found in (a). Simplify your answer as much as possible.

(c) [32 marks] Repeat (b), but now let γ enclose only z_1 and z_2. [Hint: can you see how to use the Cauchy integral theorem to replace γ with two small circles around z_1 and z_2?]

(d) [32 marks] Repeat (b), but now let γ enclose all four points.

(e) [24 marks] Now let C be any simple (non-selfintersecting) piecewise-smooth curve whose endpoints are any two of the points z_1, z_2, z_3, z_4, which also passes through the remaining two points [Hint: it will be useful to have it start at z_1 and go to z_2 next], and which crosses the real axis exactly once, at some point $x_0 > 0$. Draw this curve on the plane you drew in (a). Let $D = \mathbf{C}\backslash C$ denote the complex plane with the curve C removed. Use your result from (d), together with the Cauchy integral theorem if necessary, to show that if we require the contour in the definition of f to be strictly within D, then f becomes a single-valued function.

(f) [16 marks] Using the single-valued version of f described in (e), calculate

$$\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z),$$

where $z \to x_0^{\pm}$ means that z approaches from the right ($+$) or left ($-$) of the curve C. [Hint: can you see how to apply your result from (c)?] Does this difference of limits depend on your choice of C? (You do not need to give a justification.)

[2(a) and 3(f) correspond to Quiz 2.]