1. (a) [16 marks] Find all complex numbers z which satisfy the equation $z^4 - z^2 + 1 = 0$, and plot them on the complex plane. [Hint: remember the quadratic formula: $az^2 + bz + c = 0$ has solutions

$$z = \frac{1}{2a}\left(-b + (b^2 - 4ac)^{1/2}\right),$$

where the square root denotes the full (multi-valued) complex square root.]

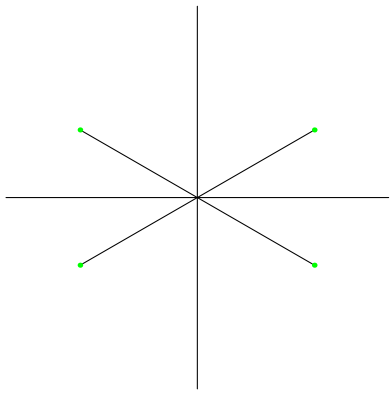
We may apply the quadratic formula to this equation by making the substitution $u = z^2$; then $u^2 - u + 1 = 0$, so

$$u = \frac{1}{2}\left(1 + (1-4)^{1/2}\right) = \frac{1}{2}(1 \pm i\sqrt{3}) = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{\pm i\pi/3}.$$

Now $u = z^2$, so we obtain the four solutions

$$z = e^{\pm i\pi/6},\ e^{\pm i(\pi/6+\pi)} = e^{i\pi/6},\ e^{5i\pi/6},\ e^{7i\pi/6},\ e^{11i\pi/6}.$$

These can be drawn as follows:



[2 marks for each correct expression for a root, 2 marks for each correct plotted root.]

(b) [16 marks] Write $z = x + iy$, expand out z^4, and use the Cauchy-Riemann equations to show that it is analytic at all points in the complex plane. [This part corresponds to Quiz 1.]

We have

$$\begin{aligned}(x+iy)^4 &= \sum_{k=0}^{4}\binom{4}{k} x^{4-k}(iy)^k = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \\ &= x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3). \qquad \text{[4 marks]}\end{aligned}$$

Letting P denote the real and Q the imaginary part of z^4, we have [2 marks/derivative, 8 marks total]

$$\frac{\partial P}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial P}{\partial y} = -12x^2y + 4y^3, \quad \frac{\partial Q}{\partial x} = 12x^2y - 4y^3, \quad \frac{\partial Q}{\partial y} = 4x^3 12xy^2,$$

so [1 mark/equation, 2 marks total]

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

and since the derivatives are continuous everywhere [2 marks], we see that z^4 is analytic on the whole complex plane.

[Marking: as indicated.]

(c) [4 marks] Use the results of (a) and (b) to find the region in the complex plane where the function

$$\frac{1}{1 - z^2 + z^4}$$

is analytic (in the sense of having a complex derivative). [You may assume without proof that z^2 is analytic.]

By the quotient rule and (b)[1 mark], as well as the fact that z^2 is analytic, $1/(1-z^2+z^4)$ will be analytic as long as the denominator is nonzero [1 mark]. By (a), this will happen when $z \neq e^{i\pi/6}$, $e^{5i\pi/6}$, $e^{7i\pi/6}$, $e^{11i\pi/6}$ [1 mark]. Thus $1/(1-z^2+z^4)$ is analytic on $\mathbf{C}\backslash\{e^{i\pi/6}, e^{5i\pi/6}, e^{7i\pi/6}, e^{11i\pi/6}\}$.[1 mark]

[Marking: as indicated.]

The following is for Question 2 and Question 3:

Now let us define a (potentially multi-valued) function f of the complex variable z by the rule

$$f(z) = \int_0^z \frac{1}{1-z'^2+z'^4}\,dz',$$

where the value of f may depend on the curve chosen from 0 to z (if so, then f will be multi-valued). If x is a real number, let

$$g(x) = \int_0^x \frac{1}{1+u^2+u^4}\,du,$$

where the integral is the usual real-variable integral; in other words, $g(x)$ is $f(x)$ where the curve defining f is required to lie along the real axis.

2. (a) [12 marks] Consider the function $f(iy)$, where the curve is taken along the imaginary axis. By parameterising this curve, show how to express $f(iy)$ in terms of $g(y)$. [Here y is an arbitrary real number.]

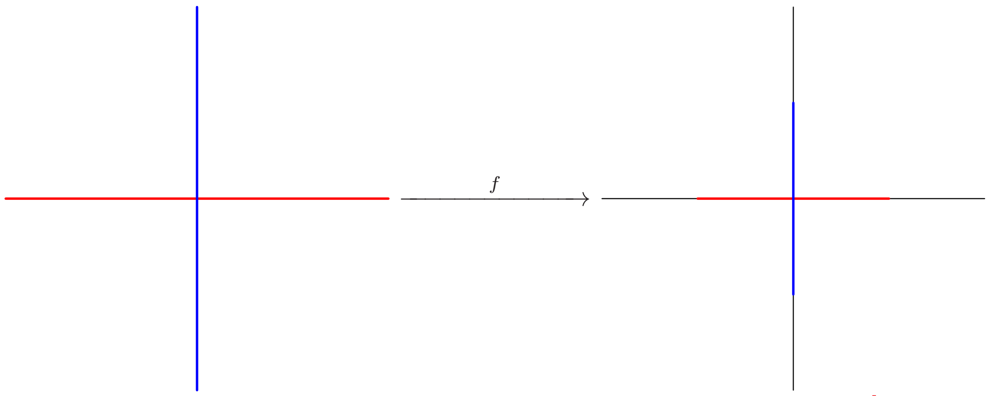
Let $y \in \mathbf{R}$, and define the curve $\gamma : [0,1] \to \mathbf{C}$ by $\gamma(t) = ity$ [2 marks]; then γ will be a curve along the imaginary axis from 0 to iy, and along this curve we have, since $\gamma'(t) = iy$[1 mark],

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \int_0^1 \frac{1}{1-(ity)^2+(ity)^4} iy\,dt\text{[2 marks]} = \int_0^1 \frac{1}{1+(ty)^2+(ty)^4} iy\,dt\text{[2 marks]}$$
$$= i\int_0^y \frac{1}{1+u^2+u^4}\,du\text{[3 marks]} = ig(y)\text{[2 marks]}.$$

[Marking: as indicated.]

(b) [12 marks] Use your result from (a) to draw the image of the real and imaginary axes under the function $z \mapsto f(z)$, where in both cases we require the curves used to lie along the respective axes. (You may assume that the function g maps the real line onto some open interval around 0.)

Evidently, along the real axis, $f(x) = \int_0^x \frac{1}{1-x^2+x^4}\,dx$ [2 marks], while from (a), $f(iy) = ig(y)$ [2 marks]. Thus f will take the real axis into some open interval around 0 on the real axis [2 marks], and the imaginary axis into some open interval around 0 on the imaginary axis [2 marks]. If we indicate this pictorially, we have



[2 marks each for correctly indicating the mapping of the real and imaginary axis]

[Marking: as indicated. Some explanation in words of the image sets should be given, but it can be less formal than what is here.]

(c) [8 marks] What is $f'(0)$? Can we conclude that f is conformal at $z=0$? Explain how this relates to your picture in (b).

Since $1/(1-z^2+z^4)$ is analytic near $z=0$ [2 marks], we will have $f'(z) = 1/(1-z^2+z^4)$ for z near 0, and in particular $f'(0) = 1/(1-0^2+0^4) = 1/1 = 1$ [2 marks]. Since f' exists and is nonzero, we see that

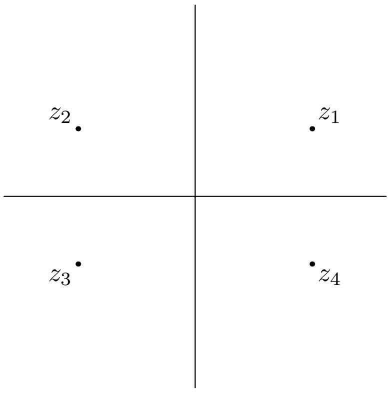
f is conformal at $z = 0$ [2 marks]. This is exemplified by the fact that the image 'curves' (in this case, line segments) in (b) make the same angle with each other as the original ones do [2 marks].

[Marking: as indicated.]

3. This is a continuation of Question 2.

(a) [8 marks] Find all points z at which f does not possess a complex derivative. (You should give a reason for your answer, but you do not need to give a full proof.) Plot these points on the complex plane. Label them z_1, z_2, z_3, z_4, in any order you wish. [Hint: it will be useful to have z_1 and z_2 lie on the same side of the real axis.]

By the fundamental theorem of calculus [2 marks] [Goursat, §31], f will have a derivative at every point at which $1/(1 - z^2 + z^4)$ does [2 marks]; hence it must have a derivative at every point except the points $e^{i\pi/6}$, $e^{5i\pi/6}$, $e^{7i\pi/6}$, $e^{11i\pi/6}$, as we discussed in 1(c) above [2 marks]. We plot these as follows: [2 marks]



[Marking: as indicated. A citation to Goursat or anywhere else is of course not required.]

In the following three parts, you are free to choose the orientation of the curve.

(b) [16 marks] Use your solution to 1(a) and your notation in (a) to factor $1 - z^2 + z^4$, and then apply the Cauchy integral formula to determine the value of the integral

$$\int_\gamma \frac{1}{1 - z^2 + z'^4}\, dz',$$

where γ is any curve enclosing z_1 but none of the other points you found in (a). Simplify your answer as much as possible.

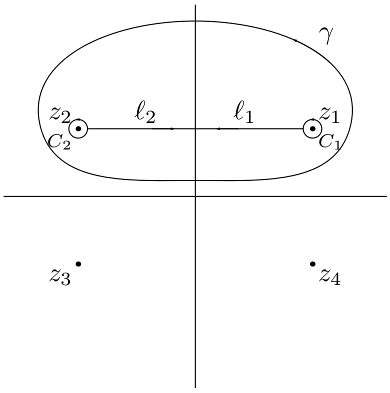
We see that we may write $1 - z^2 + z^4 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$ [4 marks]. Since γ does not enclose z_2, z_3, or z_4, the function $1/(z - z_2)(z - z_3)(z - z_4)$ will be analytic on and within γ, and hence we may apply the Cauchy integral formula to write[2 marks]

$$\begin{aligned}
\int_\gamma \frac{1}{1 - z^2 + z^4}\, dz &= \int_\gamma \frac{\frac{1}{(z-z_2)(z-z_3)(z-z_4)}}{z - z_1}\, dz && \text{[3 marks]} \\
&= \frac{2\pi i}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} && \text{[2 marks]} \\
&= \frac{2\pi i}{\sqrt{3}(\sqrt{3} + i)(i)} = \frac{2\pi i}{3 + i\sqrt{3}} \\
&= \frac{\pi\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right). && \text{[5 marks]}
\end{aligned}$$

[Marking: as indicated. The answer must be simplified to a numeric form for full marks.]

(c) [32 marks] Repeat (b), but now let γ enclose only z_1 and z_2. [Hint: can you see how to use the Cauchy integral theorem to replace γ with two small circles around z_1 and z_2?]

Consider the following picture: [2 marks for each of the curves C_1, C_2]



By the Cauchy integral theorem, we have

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \int_{C_1} \frac{1}{1-z^2+z^4}\,dz + \int_{C_2} \frac{1}{1-z^2+z^4}\,dz + \int_{\ell_1} \frac{1}{1-z^2+z^4}\,dz + \int_{\ell_2} \frac{1}{1-z^2+z^4}\,dz;$$

now the last two integrals cancel, since $1/(1-z^2+z^4)$ is continuous along the line ℓ_1 and ℓ_2, and so we have

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \int_{C_1} \frac{1}{1-z^2+z^4}\,dz + \int_{C_2} \frac{1}{1-z^2+z^4}\,dz. \qquad \text{[4 marks]}$$

But now $\int_{C_1} \frac{1}{1-z^2+z^4}\,dz = \pi\frac{1-i}{2^{3/2}}$[10 marks] by (b); and $\int_{C_2} \frac{1}{1-z^2+z^4}\,dz$ can be computed in the same way:

$$\begin{aligned}
\int_{C_2} \frac{1}{1-z^2+z^4}\,dz &= \int_{C_2} \frac{\frac{1}{(z-z_1)(z-z_3)(z-z_4)}}{z-z_2}\,dz \qquad \text{[3 marks]} \\
&= \frac{2\pi i}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\text{[2 marks]} = \frac{2\pi i}{-\sqrt{3}(i)(-\sqrt{3}+i)} \\
&= -\frac{2\pi i}{3-i\sqrt{3}} = \frac{\pi\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right), \qquad \text{[5 marks]}
\end{aligned}$$

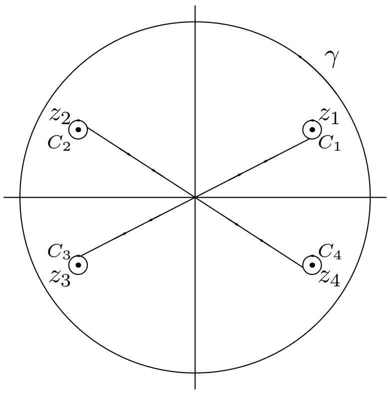
so

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \frac{\pi\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}+\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = \pi. \qquad \text{[4 marks]}$$

[Marking: as indicated. The lines ℓ_1 and ℓ_2 do not need to be used for full marks.]

(d) [32 marks] Repeat (b), but now let γ enclose all four points.

Consider the following picture: [1 mark for each curve C_1]



By the same logic as in (c), we see that

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \int_{C_1} \frac{1}{1-z^2+z^4}\,dz + \int_{C_2} \frac{1}{1-z^2+z^4}\,dz + \int_{C_3} \frac{1}{1-z^2+z^4}\,dz + \int_{C_4} \frac{1}{1-z^2+z^4}\,dz.$$

[4 marks]

We have already calculated the first two integrals; the remaining two may be calculated similarly:

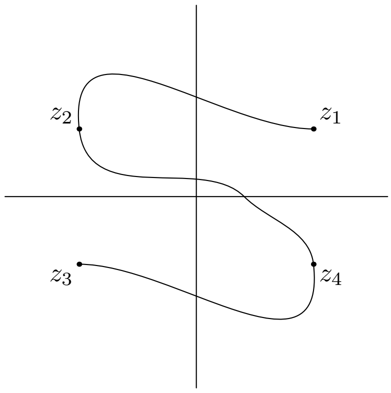
$$\int_{C_3} \frac{1}{1-z^2+z^4}\,dz = \int_{C_3} \frac{\frac{1}{(z-z_1)(z-z_2)(z-z_4)}}{z-z_3}\,dz \qquad \text{[3 marks]}$$
$$= \frac{2\pi i}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}\text{[2 marks]} = \frac{2\pi i}{-(\sqrt{3}+i)(-i)(-\sqrt{3})}$$
$$= \frac{2\pi i}{\sqrt{3}-3i} = \frac{\pi\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) \qquad \text{[5 marks]}$$
$$\int_{C_4} \frac{1}{1-z^2+z^4}\,dz = \frac{2\pi i}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}\text{[5 marks]} = \frac{2\pi i}{(-i)(\sqrt{3}-i)\sqrt{3}}$$
$$= \frac{2\pi i}{-\sqrt{3}-3i} = \frac{\pi\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right), \qquad \text{[5 marks]}$$

and we see that $\int_\gamma \frac{1}{1+z^4}\,dz = 0$.[4 marks]

[Marking: as indicated. Again, the lines between the curves do not need to be given for full marks.]

(e) [24 marks] Now let C be any simple (non-selfintersecting) piecewise-smooth curve whose endpoints are any two of the points z_1, z_2, z_3, z_4, which also passes through the remaining two points [Hint: it will be useful to have it start at z_1 and go to z_2 next], and which crosses the real axis exactly once, at some point $x_0 > 0$. Draw this curve on the plane you drew in (a). Let $D = \mathbf{C}\backslash C$ denote the complex plane with the curve C removed. Use your result from (d), together with the Cauchy integral theorem if necessary, to show that if we require the contour in the definition of f to be strictly within D, then f becomes a single-valued function.

We choose the following curve: [2 marks for each of the following: (i) initial point is one of the z_i; (ii) curve passes through all four z_i; (iii) end point is one of the z_i; (iv) curve only crosses the real axis once.]



Now let z be any point in the plane not lying on C, and let γ_1, γ_2 be two curves in $\mathbf{C}\backslash C$ from 0 to z [2 marks]. Then if any of the points z_1, z_2, z_3, z_4 lies in between γ_1 and γ_2, the whole curve C must also lie between them [6 marks]; by (d), then, the integral

$$\int_{\gamma_1-\gamma_2} \frac{1}{1-z^2+z^4}\,dz = \int_{\gamma_1} \frac{1}{1-z^2+z^4}\,dz - \int_{\gamma_2} \frac{1}{1-z^2+z^4}\,dz$$

must vanish[3 marks]. If none of the points lie between γ_1 and γ_2, then this integral will vanish by the Cauchy integral theorem[3 marks]. Thus in either case,

$$\int_{\gamma_1} \frac{1}{1-z^2+z^4}\,dz = \int_{\gamma_2} \frac{1}{1-z^2+z^4}\,dz,$$

and f will be single-valued.[2 marks]

[Marking: as indicated.]

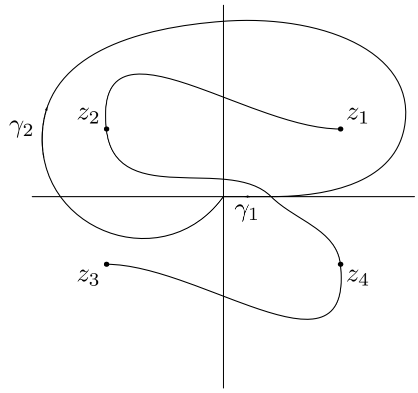
(f) [16 marks] Using the single-valued version of f described in (e), calculate

$$\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z),$$

where $z \to x_0^\pm$ means that z approaches from the right (+) or left (−) of the curve C. [Hint: can you see how to apply your result from (c)?] Does this difference of limits depend on your choice of C? (You do not need to give a justification.)

[2(a) and 3(f) correspond to Quiz 2.]

Since f is single-valued, we may use any two curves γ_1, γ_2[3 marks]. We use the following curves: [2 marks for each curve; curves should form a closed loop containing only two of the z_i – it doesn't matter which two – and not intersecting the curve C except at its intersection with the real axis]



Now $1/(1-z^2+z^4)$ is continuous at x_0, so in the limit

$$\begin{aligned}\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z) &= \int_{\gamma_2} \frac{1}{1-z^2+z^4}\,dz - \int_{\gamma_1} \frac{1}{1-z^2+z^4}\,dz \qquad \text{[2 marks]}\\ &= -\int_{\gamma_1-\gamma_2} \frac{1}{1-z^2+z^4}\,dz\text{[3 marks]} = -\pi\text{[2 marks]}\end{aligned}$$

by (c). It is quite clear that this does not depend on the choice of C, since regardless of the choice of C the curve $\gamma_1 - \gamma_2$ must enclose exactly the two points z_1 and z_2, which entirely determines the integral of $1/(1-z^2+z^4)$ over $\gamma_1 - \gamma_2$.[2 marks]

[Marking: as indicated.]