1. [9 marks] Suppose that f is a function which is analytic on the entire complex plane, and that there is a constant $C > 0$ such that $|f(z)| \le CR$ whenever $|z| = R$. If $f(0) = 0$ and $f'(0) = 1$, what is f?

We note that f may be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{[2 marks]}$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_R} \frac{f(z')}{z'^{\,n+1}}\,dz', \text{[1 mark]}$$

C_R being a circle of radius R about the origin. But now

$$|a_n| = \left|\frac{1}{2\pi i}\int_{C_R} \frac{f(z')}{z'^{\,n+1}}\,dz'\right| \le \frac{1}{2\pi}\int_{C_R} \frac{|f(z')|}{R^{n+1}}\,ds \le \frac{1}{2\pi}\int_0^{2\pi} \frac{CR}{R^{n+1}} R\,dt \text{[2 marks]} = \frac{C}{R^{n-1}} \text{[1 mark]}$$

which goes to zero as $R \to \infty$ if $n \ge 2$[1 mark]. Since a_n does not depend on R, we must have $a_n = 0$ for $n \ge 2$[1 mark]; thus $f(z) = z$[1 mark].

2. [7 marks] How many zeroes does the function $z^n e^z + \frac{1}{2}\sin z$, also on the unit disk?

We note that on the unit circle, $\frac{1}{8}|\sin z| \le \frac{1}{8}e^{|y|} \le e/8$[2 marks], while $|z^n e^z| = |e^z| = e^x \ge e^{-1} > \frac{1}{8}e$[2 marks], so that by Rouché's Theorem[1 mark], $z^n e^z + \frac{1}{8}\sin z$ has the same number of zeroes in the unit disk as $z^n e^z$[1 mark], namely n[1 mark].

3. [6 marks] Using the Taylor series for e^z around $z = 0$, find the Laurent series for $e^{1/z}$ around $z = 0$. Use this to determine $\int_C e^{1/z}\,dz$, where C is any circle centred at the origin.

We have

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \text{[1 mark]},$$

so for $z \neq 0$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \text{[1 mark]}.$$

Thus

$$\int_C e^{1/z}\,dz = \sum_{n=0}^{\infty} \frac{1}{n!}\int_C z^{-n}\,dz = 2\pi i \text{[2 marks]},$$

since only the $n = 1$ term contributes[2 marks], as we have seen many times during the course.

Evaluate the following integrals.

4. [10 marks]

$$\int_{-\infty}^{+\infty} \frac{e^{-ix}}{(x^2+4x+8)^2}\,dx.$$

Since we have a factor of e^{-ix} in the numerator, we must close in the lower half-plane. We will use the contour shown in the figure[1 mark]. Now $z^2 + 4z + 8 = 0$ gives $z = -2 + \frac{1}{2}(16 - 32)^{1/2} = -2 \pm 2i$[1 mark], so in the lower half-plane we have only one pole, at $-2 - 2i$[1 mark]. Now since for R sufficiently large we have

$$\frac{1}{|z^2+4z+8|^2} \le \frac{1}{(R^2-4R-8)^2}$$

and this goes to zero as $R \to \infty$[1 mark], we have

$$\int_{C_R} \frac{e^{-iz}}{(z^2+4z+8)^2}\,dz \to 0$$

as $R \to \infty$ by the Jordan Lemma[1 mark] applied on the lower half-plane. Thus

$$\begin{aligned}
\lim_{R\to\infty} \int_{L_R} \frac{e^{-ix}}{(x^2+4x+8)^2}\,dx &= -2\pi i \operatorname{Res}_{-2-2i} \frac{e^{-iz}}{(z^2+4z+8)^2} \text{[1 mark]} \\
&= -2\pi i \left.\frac{d}{dz}\frac{e^{-iz}}{(z+2-2i)^2}\right|_{z=-2-2i} \text{[1 mark]} \\
&= -2\pi i \left.\left[\frac{-ie^{-iz}}{(z+2-2i)^2} - 2\frac{e^{-iz}}{(z+2-2i)^3}\right]\right|_{z=-2-2i} \text{[1 mark]} \\
&= \frac{2\pi i}{-16}\left[ie^{2i-2} + 2\frac{e^{2i-2}}{-4i}\right] \text{[1 mark]} \\
&= -\frac{\pi i}{8}e^{2i-2}\left[\frac{3}{2}i\right] = \frac{3\pi}{16}e^{2i-2} \text{[1 mark]}.
\end{aligned}$$

5.

[15 marks]

$$\int_0^{+\infty} \frac{\cos x^4 - (1+\sqrt{2})\sin x^4}{1+x^8}\,dx.$$

We follow the method used in the homework and work with

$$\int_0^{+\infty} \frac{e^{ix^4}}{1+x^8}\,dx \text{[1 mark]}.$$

We wish to close along a wedge; we will choose the angle θ so that z^4 is in the upper half plane for $\arg z \in [0, \theta]$ while $\left(e^{i\theta}z\right)^8 = z^8$[1 mark]. This last gives $e^{8i\theta} = 1$, or $\theta = n\pi/4$, while the former requires $\theta \le \pi/4$; thus we take $\theta = \pi/4$[1 mark]. Thus we close using the contour shown in the figure.[1 mark]

Now

$$\int_{C_R} \frac{e^{iz^4}}{1+z^8}\,dz = \int_0^{\pi/4} \frac{e^{ie^{4it}R^4}}{1+R^8e^{8it}} iRe^{it}\,dt \text{[1 mark]}.$$

For $t \in [0, \pi/4]$, e^{4it} will have a nonnegative imaginary part; in particular, $\operatorname{Im} e^{4it} = \sin 4t \ge \frac{2}{\pi}4t$ for $t \in [0, \pi/8]$ by the Jordan inequality[1 mark]. Thus

$$\begin{aligned}
\left|\int_0^{\pi/4} \frac{e^{ie^{4it}R^4}}{1+R^8e^{8it}}\,dt\right| &\le \int_0^{\pi/4} \frac{e^{-R^4\sin 4t}}{R^8-1}\,dt = 2\int_0^{\pi/8} \frac{e^{-R^4\sin 4t}}{R^8-1}\,dt \text{[2 marks]} \\
&\le 2\int_0^{\pi/8} \frac{e^{-\frac{8}{\pi}R^4 t}}{R^8-1}\,dt \le \frac{\pi}{4R^4(R^8-1)}\left(1-e^{-R^4}\right) \to 0 \text{[1 mark]}
\end{aligned}$$

as $R \to \infty$. Further,

$$\int_{L_R'} \frac{e^{iz^4}}{1+z^8}\,dz = -\int_0^R \frac{e^{-it^4}e^{i\pi/4}}{1+t^8}\,dt \text{[1 mark]},$$

so (note that $z^8 + 1 = 0$ gives $z = e^{i\pi/8 + n\pi/4}$, so inside the wedge we have only one pole, at $e^{i\pi/8}$)

$$2\pi i \operatorname{Res}_{e^{i\pi/8}} \frac{e^{iz^4}}{1+z^8} = \lim_{R\to\infty}\left(\int_{L_R} \frac{e^{iz^4}}{1+z^8}\,dz + \int_{L_R'} \frac{e^{iz^4}}{1+z^8}\,dz\right) = \lim_{R\to\infty}\int_0^R \frac{e^{it^4} - e^{i\pi/4}e^{-it^4}}{1+t^8}\,dt \text{[1 mark]}.$$

Now

$$\begin{aligned}
e^{it^4} - e^{i\pi/4}e^{-it^4} &= \cos t^4 + i\sin t^4 - \left(\frac{1}{\sqrt{2}}\cos t^4 + \frac{1}{\sqrt{2}}\sin t^4 + i\left[-\frac{1}{\sqrt{2}}\sin t^4 + \frac{1}{\sqrt{2}}\cos t^4\right]\right) \\
&= \left(1-\frac{1}{\sqrt{2}}\right)\cos t^4 - \frac{1}{\sqrt{2}}\sin t^4 + i\left[\left(1+\frac{1}{\sqrt{2}}\right)\sin t^4 - \frac{1}{\sqrt{2}}\cos t^4\right] \\
&= \left(1-\frac{1}{\sqrt{2}}\right)\left[\cos t^4 - (\sqrt{2}+1)\sin t^4\right] + i\frac{1}{\sqrt{2}}\left[\left(\sqrt{2}+1\right)\sin t^4 - \cos t^4\right] \text{[1 mark]}
\end{aligned}$$

so the integral we want is $1/(1-1/\sqrt{2}) = \frac{1}{2}(1+\frac{1}{\sqrt{2}})$ times the real part of the above limit. Now the residue may be computed as follows:

$$\mathrm{Res}_{e^{i\pi/8}} \frac{e^{iz^4}}{1+z^8} = \frac{e^{ie^{i\pi/2}}}{8e^{7i\pi/8}} = \frac{1}{8e}\left[\cos\frac{7\pi}{8} - i\sin\frac{7\pi}{8}\right] \text{[2 marks]},$$

and we have finally

$$\int_0^\infty \frac{\cos x^4 - (1+\sqrt{2})\sin x^4}{1+x^8}\,dx = \frac{\pi}{8e}\sin\frac{7\pi}{8}\left(1+\frac{1}{\sqrt{2}}\right) \text{[1 mark]}.$$

6.

[20 marks]

$$\int_0^{+\infty} \frac{x^{-\alpha}}{x^4+3x^2+2}\,dx,$$

where $\alpha \in (0,1)$ and the exponential denotes the standard version of this function on positive real numbers. [Hint: while this can be done with a keyhole contour, there is another contour which requires fewer computations.]

We will close on an indented contour wedge. We pick the angle in the same fashion as in 5; thus we want $e^{2i\theta} = 1$, so $\theta = n\pi$, and we take $\theta = \pi$[2 marks] to minimise the number of poles within the contour. Thus we have the contour in the figure[1 mark]. We shall take the branch of the exponential function with a cut along the negative imaginary axis and an angle between $-\pi/2$ and $3\pi/2$.

Now $z^4 + 3z^4 + 2 = 0$ gives $z^2 = -\frac{3}{2} \pm \frac{1}{2} = -2, -1$, so $z = \pm i\sqrt{2}, \pm i$[2 marks], and we have only $z = i\sqrt{2}, i$ within the contour[1 mark]. At these points we have the residues

$$\mathrm{Res}_i \frac{z^{-\alpha}}{z^4+3z^2+2} = \mathrm{Res}_i \frac{z^{-\alpha}}{(z^2+2)(z^2+1)} = \frac{i^{-\alpha}}{1\cdot 2i} = \frac{e^{-i\alpha\frac{\pi}{2}}}{2i} = -\frac{i}{2}e^{-i\alpha\frac{\pi}{2}} \text{[2 marks]},$$

$$\mathrm{Res}_{i\sqrt{2}} \frac{z^{-\alpha}}{z^4+3z^2+2} = \frac{(i\sqrt{2})^{-\alpha}}{2i\sqrt{2}(-1)} = \frac{2^{-\alpha/2}e^{-i\alpha\frac{\pi}{2}}}{-2i\sqrt{2}} = i2^{-(\alpha+3)/2}e^{-i\alpha\pi/2} \text{[2 marks]}.$$

Further, we claim that

$$\int_{C_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz,\ \int_{C'_\epsilon} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz \to 0 \quad \text{as} \quad R\to\infty, \epsilon\to 0^+.$$

We have

$$\left|\int_{C_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz\right| = \left|\int_0^\pi \frac{R^{-\alpha}e^{-i\alpha t}}{R^4e^{4it}+3R^2e^{2it}+2}\,ds\right| \le \int_0^\pi \frac{R^{-\alpha}}{R^4-3R^2-2}R\,dt$$
$$\le \frac{R^{1-\alpha}\pi}{R^4-3R^2-2} \to 0 \text{ as } R\to\infty \text{[2 marks]},$$
$$\left|\int_{C'_\epsilon} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz\right| = \left|\int_0^\pi \frac{\epsilon^{-\alpha}}{\epsilon^4e^{4it}+3\epsilon^2e^{2it}+2}\epsilon\,dt\right| \le \frac{\epsilon^{1-\alpha}\pi}{2-3\epsilon^2-\epsilon^4} \to 0 \text{ as } \epsilon\to 0^+ \text{[2 marks]}.$$

Thus we will have

$$\lim_{\epsilon\to 0^+}\lim_{R\to\infty} \int_{L'_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz + \int_{L_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz = 2\pi i\left[-\frac{i}{2}e^{-i\alpha\frac{\pi}{2}} + i2^{-(\alpha+3)/2}e^{-i\alpha\pi/2}\right] \text{[2 marks]}.$$

Now (since $-L'_R$ can be parameterised by $-t$, $t\in[\epsilon, R]$)

$$\int_{L_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz = \int_\epsilon^R \frac{(-t)^{-\alpha}}{t^4+3t^2+2}\,dt = \int_\epsilon^R \frac{e^{-i\alpha\pi}t^{-\alpha}}{t^4+3t^2+2}\,dt = e^{-i\alpha\pi}\int_{L_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz \text{[1 mark]};$$

thus we have

$$\begin{aligned}\int_0^\infty \frac{x^{-\alpha}}{x^4+3x^2+2}\,dx &= \frac{2\pi i}{1+e^{-i\alpha\pi}}\left[-\frac{i}{2}e^{-i\alpha\frac{\pi}{2}} + i2^{-(\alpha+3)/2}e^{-i\alpha\pi/2}\right] \text{[1 mark]}\\ &= \frac{\pi}{2\cos\frac{1}{2}\alpha\pi}\left[1-2^{-(\alpha+1)/2}\right] \text{[2 marks]}.\end{aligned}$$

7. [11 marks] Use the fact that $1/(z+1)^2$ is analytic on the right half-plane to solve the following problem on the wedge $D = \{(x,y)\,|\,x>0,\ -x\leq y\leq x\}$:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = -\frac{4x^2}{(1+4x^4)^2}.$$

We see that, writing $z = x+iy$,

$$\begin{aligned}\frac{1}{(1+z)^2} = \frac{1}{[(1+x)+iy]^2} &= \frac{1}{(1+x)^2-y^2+2i(1+x)y}\\ &= \frac{(x+1)^2-y^2-2i(1+x)y}{[(1+x)^2-y^2]^2+4(1+x)^2y^2} = \frac{(x+1)^2-y^2-2i(1+x)y}{[(1+x)^2+y^2]^2}.\text{[2 marks]}\end{aligned}$$

Now let us try to convert the original problem to one on the right half-plane by using the map $z\mapsto z^{1/2}$[2 marks]. The boundary conditions will transform as follows:

$$\begin{aligned}v|_{x=0,y\geq 0} &= u|_{y=x}(\sqrt{x/2},\sqrt{x/2}) = -\frac{2y}{(1+y^2)^2}, \text{[2 marks]}\\ v|_{x=0,y\leq 0} &= u|_{y=-x}(\sqrt{x/2},-\sqrt{x/2}) = \frac{2y}{(1+y^2)^2}, \text{[2 marks]}\end{aligned}$$

so

$$v = -\frac{2(1+x)y}{[(1+x)^2+y^2]^2}\text{[2 marks]},$$

and

$$u = v\circ z^2 = v(x^2-y^2, 2xy) = -\frac{4(1+x^2-y^2)xy}{[(1+x^2-y^2)^2+4x^2y^2]^2}$$

is the desired solution.[1 mark]

[It was not until after the marking was commenced that the mistake in the above problem was discovered (note that the transformed initial data should have been the same thing on both $y\geq 0$ and $y\leq 0$, which would require a different sign for the two half-line boundaries in the original initial data). The marking was carried out in such a way as to avoid penalising anyone for this error in the problem, essentially as follows: 2 marks for working out the function $1/(1+z)^2$, 2 marks for knowing that the imaginary part of an analytic function is harmonic, 2 marks for the correct conformal transformation, 2 marks for each of the boundary conditions, and 1 mark for knowing that the final solution should be $v\circ f^{-1}$ for f the conformal map and v the solution of the transformed problem.]