Tutorial 5201 (Thursday, 5-6), Quiz 2, rubric.

It can be shown that on the set $U = \mathbf{R}^2 \backslash \{(x, 0)|x \le 0\}$ the function

$$P(x, y) = -\sqrt{(x^2 + y^2)^{\frac{1}{2}} + x}$$

is harmonic. The formula

$$Q(x_0, y_0) = \int_{(1,0)}^{(x_0,y_0)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy$$

gives a function which is harmonic on U and such that

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is analytic on U. Use this formula to determine Q on the part of the unit circle which lies in U. (The identity

$$\cos\frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

valid when $\cos\frac{x}{2}$ is nonnegative, may be useful.) What is

$$\lim_{(x,y)\to(-1,0^+)} Q(x, y) - \lim_{(x,y)\to(-1,0^-)} Q(x, y)$$

(where the limits are taken along the unit circle)? What do we call the half-line $\{(x, 0)|x \le 0\}$ in this case?

First, we note that

$$\frac{\partial P}{\partial x} = -\frac{\frac{x}{(x^2+y^2)^{1/2}} + 1}{2\sqrt{(x^2 + y^2)^{1/2} + x}}, \qquad \text{[1 mark]} \qquad \frac{\partial P}{\partial y} = -\frac{y}{2(x^2 + y^2)^{1/2}\sqrt{(x^2 + y^2)^{1/2} + x}}; \qquad \text{[1 mark]}$$

we could of course simplify the first of these but we don't really need to for this particular problem.

The unit circle can be parameterised as

$$(x(t), y(t)) = (\cos t, \sin t), \qquad \text{[1 mark]}$$

where the portion of the circle lying in U corresponds to the range $t \in (-\pi, \pi)$, and the point $(1, 0)$ corresponds to $t = 0$. Given this, we may evaluate the line integral as follows. Let $\theta \in (-\pi, \pi)$; then

$$Q(\cos\theta, \sin\theta) = \int_{(1,0)}^{(\cos\theta,\sin\theta)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy.$$

Now at the point $(\cos t, \sin t)$ on the unit circle we have

$$\frac{\partial P}{\partial x} = -\frac{\cos t + 1}{2\sqrt{\cos t + 1}}, \qquad \frac{\partial P}{\partial y} = -\frac{\sin t}{2\sqrt{1 + \cos t}}, \qquad \text{[0.5 marks each]}$$

so this integral becomes

$$\begin{aligned}
Q(\cos\theta, \sin\theta) &= \int_0^\theta \frac{\sin t}{2\sqrt{1 + \cos t}}(-\sin t) - \frac{\cos t + 1}{2\sqrt{\cos t + 1}} \cos t\, dt && \text{[1 mark for } -\sin t \text{ and } \cos t\text{, 1 mark for bounds]} \\
&= \int_0^\theta \frac{-\sin^2 t - \cos^2 t - \cos t}{2\sqrt{1 + \cos t}}\, dt \\
&= \int_0^\theta -\frac{\sqrt{1 + \cos t}}{2}\, dt = \int_0^\theta -\frac{1}{\sqrt{2}} \cos\frac{t}{2}\, dt && \text{[1 mark]} \\
&= -\sqrt{2} \sin\frac{t}{2}\Big|_0^\theta = -\sqrt{2}\sin\frac{\theta}{2}, && \text{[1 mark]}
\end{aligned}$$

the desired answer. (Note that we are allowed to write $\cos\frac{t}{2} = \sqrt{\frac{1+\cos t}{2}}$, taking the positive square root, since $t \in (-\pi, \pi)$ implies that $\cos\frac{t}{2} > 0$. It is instructive to think about this last statement geometrically.)

Penultimately, then, we have, taking the limits along the unit circle, [1 mark for π^-, 1 mark for π^+, 1 mark for the final answer]

$$\begin{aligned}\lim_{(x,y)\to(-1,0^+)} Q(x,y) - \lim_{(x,y)\to(-1,0^-)} Q(x,y) &= \lim_{\theta\to\pi^-} Q(\cos\theta, \sin\theta) - \lim_{\theta\to-\pi^+} Q(\cos\theta, \sin\theta) \\ &= \lim_{\theta\to\pi^-} -\sqrt{2}\sin\frac{\theta}{2} - \lim_{\theta\to-\pi^+} -\sqrt{2}\sin\frac{\theta}{2} = -\sqrt{2} - \sqrt{2} = -2\sqrt{2}.\end{aligned}$$

In this case, the function P is (up to a constant multiple) the real part of the branch of the square-root function corresponding to the given set, and the half-line $\{(x,0)|x \leq 0\}$ is the associated branch cut. [1 mark for saying something related to branch cut or branch point]