Tutorial 5101 (Tuesday, 5-6), Quiz 2, rubric.

It can be shown that on the set $U = \mathbf{R}^2 \backslash \{(x,0)|x \geq 0\}$, i.e., the plane with the positive x axis and origin removed, the function

$$P(x,y) = \sqrt{(x^2+y^2)^{\frac{1}{2}} - x}$$

is harmonic. The formula

$$Q(x_0,y_0) = \int_{(1,0)}^{(x_0,y_0)} -\frac{\partial P}{\partial y}\,dx + \frac{\partial P}{\partial x}\,dy$$

gives a function which is harmonic on U and such that

$$f(x+iy) = P(x,y) + iQ(x,y)$$

is analytic on U. Use this formula to determine Q on the part of the unit circle which lies in U. (The identity

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}},$$

valid when $\sin\frac{x}{2}$ is nonnegative, may be useful.) What is

$$\lim_{(x,y)\to(1,0^+)} Q(x,y) - \lim_{(x,y)\to(1,0^-)} Q(x,y)$$

(where the limits are taken along the unit circle)? What do we call the half-line $\{(x,0)|x \geq 0\}$ in this case?

First, we note that

$$\frac{\partial P}{\partial x} = \frac{\frac{x}{(x^2+y^2)^{1/2}} - 1}{2\sqrt{(x^2+y^2)^{1/2} - x}}, \quad \text{[1 mark]} \qquad \frac{\partial P}{\partial y} = \frac{y}{2(x^2+y^2)^{1/2}\sqrt{(x^2+y^2)^{1/2} - x}}; \quad \text{[1 mark]}$$

we could of course simplify the first of these but we don't really need to for this particular problem.

The unit circle can be parameterised as

$$(x(t), y(t)) = (\cos t, \sin t), \qquad \text{[1 mark]}$$

where the portion of the circle lying in U corresponds to the range $t \in (0, 2\pi)$, and the point $(1,0)$ corresponds to $t = 0$. (That this value for t is not in the range just given merely means that the function Q is not defined at the point $(1,0)$; it does not affect any other part of the problem.) Given this, we may evaluate the line integral as follows. Let $\theta \in (0, 2\pi)$; then

$$Q(\cos\theta, \sin\theta) = \int_{(1,0)}^{(\cos\theta,\sin\theta)} -\frac{\partial P}{\partial y}\,dx + \frac{\partial P}{\partial x}\,dy.$$

Now at the point $(\cos t, \sin t)$ on the unit circle we have

$$\frac{\partial P}{\partial x} = \frac{\cos t - 1}{2\sqrt{1-\cos t}}, \qquad \frac{\partial P}{\partial y} = \frac{\sin t}{2\sqrt{1-\cos t}}, \qquad \text{[0.5 marks each]}$$

so this integral becomes

$$\begin{aligned} Q(\cos\theta, \sin\theta) &= \int_0^\theta -\frac{\sin t}{2\sqrt{1-\cos t}}(-\sin t) + \frac{\cos t - 1}{2\sqrt{1-\cos t}}\cos t\,dt && \text{[1 mark for } -\sin t \text{ and } \cos t\text{, 1 mark for bounds]} \\ &= \int_0^\theta \frac{\sin^2 t + \cos^2 t - \cos t}{2\sqrt{1-\cos t}}\,dt \\ &= \int_0^\theta \frac{\sqrt{1-\cos t}}{2}\,dt = \int_0^\theta \frac{1}{\sqrt{2}}\sin\frac{t}{2}\,dt && \text{[1 mark]} \\ &= -\sqrt{2}\cos\frac{t}{2}\Big|_0^\theta = \sqrt{2}\left(1 - \cos\frac{\theta}{2}\right), && \text{[1 mark]} \end{aligned}$$

the desired answer. (Note that we are allowed to write $\sin\frac{t}{2} = \sqrt{\frac{1-\cos t}{2}}$, taking the positive square root, since $t \in (0, 2\pi)$ implies that $\sin\frac{t}{2} > 0$. It is instructive to think about this last statement geometrically.)

Penultimately, then, we have, taking the limits along the unit circle, [1 mark for $\theta \to 0^+$, 1 mark for $\theta \to 2\pi^-$, 1 mark for the final answer]

$$\begin{aligned}\lim_{(x,y)\to(1,0^+)} Q(x,y) - \lim_{(x,y)\to(1,0^-)} Q(x,y) &= \lim_{\theta\to0^+} Q(\cos\theta, \sin\theta) - \lim_{\theta\to2\pi^-} Q(\cos\theta, \sin\theta) \\ &= \lim_{\theta\to0^+} \sqrt{2}\left(1 - \cos\frac{\theta}{2}\right) - \lim_{\theta\to2\pi^-} \sqrt{2}\left(1 - \cos\frac{\theta}{2}\right) = 0 - \sqrt{2}(1-(-1)) = -2\sqrt{2}.\end{aligned}$$

In this case, the function P is (up to a constant multiple) the imaginary part of the branch of the square-root function corresponding to the given set, and the half-line $\{(x,0)|x \geq 0\}$ is the associated branch cut. [1 mark for saying something related to branch cut or branch point]