Tutorial 5101 (Tuesday, 5-6), Quiz 1, rubric.

1. Use the Cauchy-Riemann equations to determine whether the following function is analytic on the region $x, y > 0$:

$$f(x+iy) = -\log(x^2+y^2) - 2i \arctan \frac{y}{x}.$$

Recall that the derivative of the arctangent function is given by $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$.

Solution: Let $P = -\log(x^2+y^2)$ denote the real and $Q = -2 \arctan \frac{y}{x}$ the imaginary part of f, respectively. Then we see that

$$\frac{\partial P}{\partial x} = -\frac{2x}{x^2+y^2}, \qquad \frac{\partial P}{\partial y} = -\frac{2y}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{2\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{2y}{x^2+y^2}, \qquad \frac{\partial Q}{\partial y} = -2\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = -\frac{2x}{x^2+y^2},$$

from which it is clear that

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

and

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

i.e., that the Cauchy-Riemann equations are satisfied. Since the partial derivatives are also continuous on $x, y > 0$, the function must be analytic.

Marking: 1 mark for each partial derivative, 1 mark for each of the Cauchy-Riemann equations, for 6 marks total. Mentioning continuity is *not* required since the problem was ambiguous as to whether that was needed or not, and since the other version of this problem does not require saying anything about continuity.

2. Use the Cauchy-Riemann equations to determine whether the following function is analytic on the region $x, y > 0$:

$$f(x+iy) = -3\log(x^2+y^2) - 6i \arctan \frac{x}{y}.$$

(Recall that the derivative of the arctangent function is given by $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$.)

Solution: Let $P = -3\log(x^2+y^2)$ denote the real and $Q = -6 \arctan \frac{x}{y}$ the imaginary part of f, respectively. Then we see that

$$\frac{\partial P}{\partial x} = -\frac{6x}{x^2+y^2}, \qquad \frac{\partial P}{\partial y} = -\frac{6y}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = -6\frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2} = -\frac{6y}{x^2+y^2}, \qquad \frac{\partial Q}{\partial y} = -6\frac{-\frac{x}{y^2}}{1+\left(\frac{x}{y}\right)^2} = \frac{6x}{x^2+y^2},$$

from which we see that

$$\frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} \neq -\frac{\partial Q}{\partial x},$$

so that the function cannot be analytic as the Cauchy-Riemann equations are not satisfied.

Marking: 1 mark for each partial derivative, 2 marks for concluding the Cauchy-Riemann equations are not satisfied (at least one of them has to be written out or indicated).