Tutorial 0301 (Wednesday, 10-11), Quiz 1, rubric.

1. Use the Cauchy-Riemann equations to determine whether the following function is analytic on the complex plane:

$$\cos(x^2 - y^2)\cosh(2xy) - i\sin(x^2 - y^2)\sinh(2xy)$$

Solution: Let $P = \cos(x^2 - y^2)\cosh(2xy)$ and $Q = -\sin(x^2 - y^2)\sinh(2xy)$ be the real and imaginary parts of the function, respectively. Then we have

$$\begin{aligned}\frac{\partial P}{\partial x} &= -2x\sin(x^2 - y^2)\cosh(2xy) + 2y\cos(x^2 - y^2)\sinh(2xy),\\ \frac{\partial P}{\partial y} &= 2y\sin(x^2 - y^2)\cosh(2xy) + 2x\cos(x^2 - y^2)\sinh(2xy),\\ \frac{\partial Q}{\partial x} &= -2x\cos(x^2 - y^2)\sinh(2xy) - 2y\sin(x^2 - y^2)\cosh(2xy),\\ \frac{\partial Q}{\partial y} &= 2y\cos(x^2 - y^2)\sinh(2xy) - 2x\sin(x^2 - y^2)\cosh(2xy),\end{aligned}$$

from which we see that

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

and

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

so that the Cauchy-Riemann equations are satisfied and the function is therefore analytic as the partial derivatives of P and Q are continuous.

Marking: 1 mark for each partial derivative, 1 mark for each of the Cauchy-Riemann equations, for 6 marks total. Mentioning continuity is *not* required since the problem was ambiguous as to whether that was needed or not.

2. Use the Cauchy-Riemann equations to determine whether the following function is analytic on the complex plane:

$$\cos(2xy)\cosh(x^2 - y^2) + i\sin(2xy)\sinh(x^2 - y^2)$$

Solution: Let $P = \cos(2xy)\cosh(x^2 - y^2)$ and $Q = \sin(2xy)\sinh(x^2 - y^2)$ be the real and imaginary parts of the function, respectively. Then we have

$$\begin{aligned}\frac{\partial P}{\partial x} &= -2y\sin(2xy)\cosh(x^2 - y^2) + 2x\cos(2xy)\sinh(x^2 - y^2),\\ \frac{\partial P}{\partial y} &= -2x\sin(2xy)\cosh(x^2 - y^2) - 2y\cos(2xy)\sinh(x^2 - y^2),\\ \frac{\partial Q}{\partial x} &= 2y\cos(2xy)\sinh(x^2 - y^2) + 2x\sin(2xy)\cosh(x^2 - y^2),\\ \frac{\partial Q}{\partial y} &= 2x\cos(2xy)\sinh(x^2 - y^2) - 2y\sin(2xy)\cosh(x^2 - y^2),\end{aligned}$$

from which we see that

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

and

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

so that the Cauchy-Riemann equations are satisfied and the function is therefore analytic as the partial derivatives of P and Q are continuous.

Marking: 1 mark for each partial derivative, 1 mark for each of the Cauchy-Riemann equations, for 6 marks total. Mentioning continuity is *not* required since the problem was ambiguous as to whether that was needed or not.