Tutorial 0201 (Thursday, 4-5), Quiz 1, rubric.

1. Use the Cauchy-Riemann equations to determine whether the following function is analytic on the region $x, y \neq 0$:

$$f(x+iy) = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2}$$

Solution: Let $P = \frac{y}{x^2+y^2}$ denote the real and $Q = \frac{x}{x^2+y^2}$ the imaginary part of f, respectively. Then we see that

$$\frac{\partial P}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial P}{\partial y} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2},$$
$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial Q}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

from which it is clear that

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

and

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

i.e., that the Cauchy-Riemann equations are satisfied. Since the partial derivatives are also continuous on $x, y \neq 0$, the function must be analytic.

Marking: 1 mark for each partial derivative, 1 mark for each of the Cauchy-Riemann equations, for 6 marks total. Mentioning continuity is *not* required since the problem was ambiguous as to whether that was needed or not.

2. Use the Cauchy-Riemann equations to determine whether the following function is analytic on the region $x, y \neq 0$:

$$f(x+iy) = -\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}.$$

Solution: Let $P = -\frac{x}{x^2+y^2}$ denote the real and $Q = \frac{y}{x^2+y^2}$ the imaginary part of f, respectively. Then we see that

$$\frac{\partial P}{\partial x} = -\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}, \qquad \frac{\partial P}{\partial y} = \frac{2xy}{(x^2+y^2)^2},$$
$$\frac{\partial Q}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial Q}{\partial y} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2},$$

from which we see that

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

i.e., that the Cauchy-Riemann equations are satisfied. Since the partial derivatives are also continuous on $x, y \neq 0$, the function must be analytic.

Marking: 1 mark for each partial derivative, 1 mark for each of the Cauchy-Riemann equations, for 6 marks total. Mentioning continuity is *not* required since the problem was ambiguous as to whether that was needed or not.