## PRACTICE PROBLEMS. MULTIVARIABLE CALCULUS AND LINEAR ALGEBRA.

1. Recall that a circle of radius $r$ centred at a point $(x_0, y_0)$ oriented (or we often say *traversed*) counterclockwise can be parameterised by

$$x(t) = r\cos t + x_0, \qquad y(t) = r\sin t + y_0. \qquad (t \in [0, 2\pi])$$

Use this parametrisation and the definition of arclength to compute the length of the circle of radius 2 centred at the point $(-3, 4)$. Does the integral depend on the centre point?

2. Recall that the graph of a function $y = f(x)$ on an interval $[a, b]$ can be parameterised by

$$x(t) = t, \qquad y(t) = f(t). \qquad (t \in [a, b])$$

Use this to calculate the length of the parabolic arch $y = 1 - x^2$ on the interval $[-1, 1]$. [Hint: trigonometric substitution!]

3. Consider the two vector fields

$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\left(x\mathbf{i} + y\mathbf{j}\right), \qquad \mathbf{G}(x, y) = \frac{1}{x^2 + y^2}\left(-y\mathbf{i} + x\mathbf{j}\right),$$

defined on $\mathbf{R}^2 \backslash (0, 0)$.

(a) Calculate the divergence and curl of these vector fields on their domain. (Recall that the *divergence* of a two-dimensional vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is given by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.)$$

(b) Using the parameterisation of a circle given in problem 1, calculate the following line integrals:

$$\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{x}, \qquad \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{x}, \qquad \int_{\gamma_1} \mathbf{G} \cdot d\mathbf{x}, \qquad \int_{\gamma_2} \mathbf{G} \cdot d\mathbf{x},$$

where $\gamma_1$ is the upper and $\gamma_2$ the lower unit semicircle, both traversed counterclockwise.

(c) From (b), find the line integrals

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{x}, \qquad \int_{\gamma} \mathbf{G} \cdot d\mathbf{x},$$

where $\gamma = \gamma_1 \cup \gamma_2$ is the full unit circle traversed counterclockwise. Given your results from (a), do your results contradict Green's theorem? Why or why not?

[As some of you may recognise, these vector fields arise from the complex function $z \mapsto \frac{1}{z}$, and the line integrals above give an example of Cauchy's integral formula, which we shall study in great detail later on in the course.]

4. Define the following functions:

$$f(x, y) = \frac{\cosh x \cos y}{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}, \qquad g(x, y) = -\frac{\sinh x \sin y}{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y},$$

and compute the following partial derivatives:

$$\frac{\partial}{\partial x} f(x, y), \qquad \frac{\partial}{\partial y} f(x, y), \qquad \frac{\partial}{\partial x} g(x, y), \qquad \frac{\partial}{\partial y} g(x, y).$$

Are any of these equal (or almost equal)?

5. Repeat problem 4 for the functions

$$f(x, y) = \log\sqrt{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}, \qquad g(x, y) = \tan^{-1}\frac{\sinh x \sin y}{\cosh x \cos y},$$

where (as will be usual throughout this course) log denotes the natural logarithm.

6. Let $\gamma$ denote the graph of $y = \sin x$, parameterised as in problem 2, on the interval $[0, 4\pi]$. If

$$\mathbf{F}(x, y) = x\mathbf{i} + \left(e^{\sqrt{1-\cos^2 x}} + y^5 - 10y^4 + 8y^3 - 6y^2 + 2\right)\mathbf{j},$$

calculate

$$\int_\gamma \mathbf{F}(x, y) \cdot d\mathbf{x}.$$

7. Find the inverses of the following matrices:

$$\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Can you write these matrices as products of rotations and scalings? [Except maybe the last one.]

8. Recall that if we have a map $F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(x, y) = (f(x, y), g(x, y))$, then we have the notion of the *Jacobian matrix* of $F$:

$$J(F) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

Consider the functions

$$f(x, y) = \frac{x}{x^2 + y^2}, \qquad g(x, y) = -\frac{y}{x^2 + y^2}.$$

If $F(x, y) = (f(x, y), g(x, y))$, find the Jacobian matrix of $F$ and its determinant. (If you want more practice with this, do the same thing for the functions $f$ and $g$ in problem 4.) Consider this matrix at the point $(1, 1)$. Can you write it as the product of a rotation and a scaling?