MAT334, COMPLEX VARIABLES, SUMMER 2020. MIDTERM REVIEW

So far we have covered the following topics:

- Geometry and algebra of complex numbers
- Definition of analyticity; Cauchy-Riemann equations
- Power series
- Elementary transcendetal functions (exponential, trigonometric, hyperbolic trigonometric) and their derivatives
- Roots, logarithms, and their derivatives. Branch cuts
- Inverse trigonometric functions
- Geometric interpretation of the derivative; conformal maps
- Definition of contour integrals. Specific examples
- Cauchy integral theorem. Antiderivatives and path-independence
- Cauchy integral formula
- Taylor and Laurent series

These correspond to the following sections in Goursat: 1 – 8, 11 – 14, 19, 22, 25 – 26, 28, 30 – 33, 35, 37.

Here are a few extra notes.

Multivalued functions and branch cuts.
From what we have seen so far, there are two ways in which multivalued functions can arise: (i) as inverses of functions which are not one-to-one (roots, logarithms, inverse trigonometric functions); (ii) as functions defined by contour integrals of non-analytic integrands (the function G in the long problem below, for example; note that the logarithm can also be defined in this way, as the integral of $1/z$). How we determine whether a given function is multivalued, and how we decide what kinds of branch cuts to use, depends on which case we are dealing with.

The second case is much easier to explain, so let us start there. We know that (roughly speaking) integrals of analytic functions around closed curves are always 0, which means that a function defined as the integral of an analytic function on a simply-connected region will always be single-valued. Suppose now that we have a function $f : D \to \mathbf{C}$, where D is some region in the complex plane on which f is analytic but which is not simply connected. Then we can still define a function $F : D \to \mathbf{C}$ by

$$F(z) = \int_{z_0}^{z} f(z')\, dz',$$

where $z_0 \in D$, but since the integral could now depend on the choice of curve from z_0 to z, the resulting function F will in general be multi-valued. To make F single-valued we need some way of specifying which kinds of curves we are allowed to use from z_0 to each z, in such a way that the integral will not depend on the curve chosen. The simplest way of doing this would be to choose a subset $D' \subseteq D$ of D which *is* simply-connected, and then define F only for curves lying in that subset. Such a function could then be termed a *branch* of the full function F. Something slightly similar can be done for finding branches of the logarithm, if we define it by the integral $\int_1^z 1/z'\, dz'$, though only one branch per cut results in that case and to get all the branches of Log we must do something more involved. Another, slightly more involved but more generally applicable, method is to choose a subset $D' \subseteq D$ which, while not necessarily simply-connected, is nevertheless such that the integral of f around any closed curve lying entirely in D' is still zero; this will also give a single-valued integral, which we may also term a branch of the full function F. This is the method suggested in part (k) of the long problem below.

Let us now consider the first case. This can be treated in a very general way, but we shall stick with a simpler treatment which is sufficient for the functions we have seen so far. (It would be a good idea to keep a specific function, for example a root function, in mind while reading the following discussion.) In this case there are regions U and V in the complex plane such that our function, call it g, maps U into V in a multi-valued way, and is a 'right inverse' to some other function $f : V \to U$ in the sense that the composition $f \circ g : V \to V$ is the identity map, i.e., $f(g(z)) = z$ for all $z \in V$. (For example, we could have $U = V = \mathbf{C}$, $g(z) = z^{1/2}$, $f(z) = z^2$, so $f(g(z)) = [z^{1/2}]^2 = z$ for all $z \in \mathbf{C}$.) Suppose now that we can break V up into pieces $V_1, \cdots, V_n$, which might not cover all of V, such that f is one-to-one when restricted to each V_i. Then the inverse of $f|_{V_i}$ will be a single-valued function on some subset of U, which we call a branch of g. For the functions we are dealing with, we do not need to talk about the sets V_i explicitly, and are able to specify the branch by specifying a range of angle for the points in the domain of g. To show how these two notions relate to each other, though, consider the square root function with a branch cut along the negative real axis and an interval for the angle of $(-\pi, \pi)$; this will map into the set $\{z \in \mathbf{C} | \operatorname{Re} z > 0\}$. Had we chosen the same branch cut but the range of angle $(\pi, 3\pi)$, we would get as an image set the set $\{z \in \mathbf{C} | \operatorname{Re} z < 0\}$. Thus these two sets are possible ways of breaking V up into pieces on which $z \mapsto z^2$ is one-to-one.

Here are some practice problems for review. More problems will be added.

Short problems

Basic operations

1. Perform the following operations, and sketch all complex numbers involved in the plane:

$$(1+2i)-(-3+5i), \quad 0.5\cdot(-1+i), \quad 2\cdot(-1+i), \quad 3\cdot(-1+i),$$
$$(1+i)\cdot(1-\sqrt{3}i), \quad \frac{-\sqrt{3}+i}{-1-i},$$
$$(-1+i)^{100} \quad \text{[Hint: use polar notation and exponentials!]}$$

2. Determine all possible roots of the indicated orders, and sketch them and the original complex number in the plane:

$$1^{1/2}, \quad 1^{1/4}, \quad 1^{1/5}$$
$$(-\sqrt{3}+i)^{1/5}, \quad (-\sqrt{3}-i)^{1/7}$$

3. Determine all possible values of the following exponentials:

$$i^i, \quad i^{(i^i)}, \quad (-1)^{1/100}, \quad 2^{\sqrt{2}}$$

Branch cuts

1. Describe subsets of the plane on which the following functions can be defined as single-valued analytic functions. For each function and each such set, describe all the different branches of the function by giving formulas and also describing or sketching their ranges.

$$z \mapsto z^{1/2}, \quad z \mapsto z^{2/3}, \quad z \mapsto z^i$$
$$\operatorname{Log} z$$

Long problems

1. We study the arctangent function on the complex plane.

(a) From the formulas
$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i},$$
derive a formula for $\tan z = \sin z/\cos z$ in terms of e^{iz} and e^{-iz}. Use this formula to determine a formula for $\arctan z$ where z is a complex number. For which values of z is the formula you obtain undefined?

(b) Use your formula from (a) to determine $\frac{d}{dz}\arctan z$, where it exists. For which values of z is $\arctan z$ an analytic function?

(c) From what we know about analyticity of power series, what is the maximum possible radius of convergence for the power series of $\arctan z$ around $z = 0$? How about around $z = 1$? If you find that the power series does not converge on the whole real line, does this contradict what you found in (b) about the set of z where $\arctan z$ is analytic?

(d) Are there any points in the plane about which it is not possible to expand $\arctan z$ as a Laurent series? Are there any points about which $\arctan z$ can be expanded as a Laurent series but not as a Taylor series?

(e) From your formula in (a), is $\arctan z$ a single-valued or multivalued function? If it is multivalued, how does its multivaluedness compare to that of $\arcsin z$? How about its derivative you found in (b)?

(f) Let $f(z) = \frac{d}{dz} \arctan z$ be the derivative you found in (b). Where is this function analytic? Does your answer make sense given what you found in (b) about where $\arctan z$ is analytic?

From class we know that if D is any simply-connected set on which f is analytic, then the function

$$F(z) = \int_0^z f(z')\, dz'$$

will be independent of the path from 0 to z and will satisfy $F'(z) = f(z)$. Since $f(z) = \frac{d}{dz} \arctan z$, this means that F can differ from $\arctan z$ by at most a constant. On a simply-connected region, then, F will give us (up to a constant) a single-valued version of $\arctan z$. Let us see what happens when we try to extend F to regions which are not simply-connected.

(g) Compute $\int_0^1 f(z')\, dz'$ using three different paths: (i) a path straight along the real axis; (ii) a path which loops once counterclockwise around $z = i$; (iii) a curve which loops once counterclockwise around $z = -i$. [Hint: for (ii) and (iii), draw a picture of the curve and then use it to apply the Cauchy integral formula (try factoring the denominator first!), then combine your result with (i) to get the answer.]

(h) What happens if you integrate along a path which loops around $z = i$ or $z = -i$ multiple times? Can you see how to obtain all possible values of $\arctan z$ by picking appropriate curves?

(i) Compute $\int_0^1 f(z')\, dz'$ by using a path which loops once counterclockwise around *both* $z = i$ and $z = -i$.

Now consider instead the function

$$G(z) = -\frac{\pi}{2} + \int_{-\infty}^z f(z')\, dz'.$$

As with F, G will be a single-valued antiderivative of f on any simply-connected region on which f is analytic, and hence on that region will be (a constant away from) a single-valued version of $\arctan z$. As with F, though, if I let z vary in a region which surrounds a singularity of f, then the above integral may depend on the choice of path.

(j) Repeat (g) with G in place of F. [Hint: if we integrate straight along the real axis, what is $G(0)$? Can you use this to make this part a trivial application of (g)?]

(k) Suppose now that I insist that whatever path is used to calculate G must *not* pass through the set $\{iy | y \in [-1, 1]\}$; in other words, I cut the plane along the closed line segment from $-i$ to i. Use a result similar to what you found in (i) to argue that the function G should be independent of the path for any z in the cut plane. (By 'argue that' I mean that you do not need to give a full proof, just point out the main idea; though if you can give a full proof, even better!) Calculate

$$\lim_{z \to 0^+} G(z) - \lim_{z \to 0^-} G(z),$$

where $+$ and $-$ indicate that the real part of z is positive and negative, respectively. If we let $\arctan x$ denote the standard arctangent on the real line, what is $G(x) - \arctan x$ as a function of x? [Hint: it will not be constant!]