MAT334, COMPLEX VARIABLES, SUMMER 2020. FINAL REVIEW

In the second half of the term, we have covered the following topics (or, in the case of the last three lines, will cover as much as we can):

– Taylor and Laurent series; integral formulas for coefficients.
– – Radius of convergence
– – Algebraic manipulations
– Residues and the residue theorem
– – Definition of residue
– – – Formulation in terms of contour integrals and Laurent series
– Applications to definite integrals
– – Infinite integrals: methods for closing the contour and computing residues
– – Closing the contour:
– – – Semicircle (even integrands/integrands along the whole line)
– – – Wedges:
– – – – Cases where the integral along the additional line can be evaluated directly
– – – – Cases where the integral along the additional line can be related to the original integral
– – – – – Integrals around branch cuts
– – – Indented contours (applicable to semicircle and wedge closings)
– – – Conditions on the final contour:
– – – – Integrals over added circular arcs must be zero, at least in some limit
– – – – $R|f| \to 0$, $\epsilon|f| \to 0$, Jordan's lemma
– – – – Where in the plane can we close?
– – – – – Semicircles: which half-plane
– – Methods for finding residues:
– – – Contour integral on a small circle around the pole
– – – Finding Laurent series
– – – Differentiating and taking a limit
– – – – Special case: $g(a) \neq 0$, a a simple pole of h: residue of g/h is $g(a)/h'(a)$.
– – Trigonometric integrals
– – – Which poles lie inside the contour
– Further properties of analytic functions
– – Liouville's theorem; fundamental theorem of algebra.
– – Argument principle; how $f(z)$ wraps around the origin.
– – Rouché's Theorem; fundamental theorem of algebra again.
– Harmonic functions and conformal maps
– – Poisson kernel
– – Transformations of regions and boundaries
– Analytic continuation
– – Winding around branch points; relation to branch cuts

Practice problems

[Note. These are meant for use as part of your overall review. They are not guaranteed to be comprehensive.]

Taylor and Laurent series.

1. Find $a \in \mathbf{C}$ such that

$$\frac{\cos z - a}{z^2}$$

extends to an analytic function on all of $\mathbf{C}$. What is the value of this function at $z = 0$?

2. Let a be as in 1. What is the (a) order of the pole and (b) residue at the pole at $z = 0$ of the functions

$$\frac{\cos z - a}{z^3} \quad \text{and} \quad \frac{\cos z - a}{z^4}?$$

3. By doing term-by-term integration of a geometric series, determine the Taylor series of $\operatorname{Log} z$ about $z = 1$, where here $\operatorname{Log} z$ denotes any branch of the logarithm which is defined at $z = 1$. What is the radius of convergence of this series? Does it depend on your choice of branch? [Be careful, this is tricky! You know enough to answer it right now, but the answer will probably be much clearer after we talk about analytic continuation.]

4. Determine the Laurent series of

$$\frac{\cos z}{(z-1)^2}$$

about $z = 1$ in two different ways: (a) using the formulas for the coefficients of a Laurent series in terms of contour integrals; (b) using trigonometric identities and algebraic manipulations together with the known Taylor series for $\cos z$.

5. Let $p \in (0, 1)$, and let z^p denote any branch of the indicated exponential function. Consider its Laurent series about $z = 1$. (a) Find the first three nonzero terms in the regular part of the Laurent series. (b) Are there any nonzero singular terms in this series? Why or why not? (c) What is the radius of convergence of the series?

Residues and the residue theorem.

6. Calculate the residue of

$$\frac{e^z}{(e^z-1)^n}$$

about (a) $z = 0$; (b) $z = 1$. Here n is any positive integer. [Hint: expand the numerator and denominator in power series and factor the denominator.] Repeat this problem for the function

$$\frac{e^z}{(e^z-e)^n}.$$

7. Now do the problem which is closer to what I meant the last problem to be in the first place! Calculate the residue of

$$\frac{z^{n-1}e^z}{(e^z-1)^n}$$

about (a) $z = 0$; (b) $z = 1$. Here n is any positive integer. [Hint: see last hint!] Repeat this for the function

$$\frac{z^{n-1}e^z}{(e^z-e)^n}.$$

8. If ϕ is analytic at a and $\phi(a) \neq 0$, find an expression for

$$\operatorname{Res}_a \frac{\phi(z)}{(z-a)^n}$$

in terms of a derivative of ϕ at a. [This is easy! – but also well worth noting for applications.]

9. Suppose that f has a pole of order n at $z = a$. What is the least order of derivative of $1/f$ which is not zero at a?

Integrals!
Evaluate the following integrals.

10. (a)

$$\int_{-\infty}^{\infty} \frac{1}{x^2-x+1}\,dx.$$

(b)

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2-x+1}\,dx,$$

where k is any real number. [Be careful with signs!]
(c)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 - x + 1}\, dx.$$

(d)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - x + 1)}\, dx.$$

(e – h): Repeat (a – d) with $x^2 - x + 1$ replaced by $(x^2 - x + 1)^2$.

11. (a)

$$\int_0^{\infty} \frac{1}{(x^3+1)^2}\, dx.$$

(b)

$$\int_0^{\infty} \frac{\log x}{(x^3+1)^2}\, dx.$$

(c)

$$\int_0^{\infty} \frac{x^{1/4}}{(x^3+1)^2}\, dx.$$

(d)

$$\int_0^{\infty} \frac{x^{1/4}\log x}{(x^3+1)^2}\, dx.$$

12. (a)

$$\int_0^{\infty} \frac{1}{x^6+1}\, dx.$$

(b)

$$\int_0^{\infty} \frac{\cos x^3 - \sqrt{3}\sin x^3}{x^6+1}\, dx.$$

(c)

$$\int_0^{\infty} \frac{x^{1/2}\log x}{x^6+1}\, dx.$$

[Extra (d – f): Repeat (a – c) with $x^6 + 1$ replaced by $(x^6 + 1)^2$.]

13. (a)

$$\int_0^{2\pi} \frac{1}{5 \pm 4\sin t}\, dt.$$

(b)

$$\int_0^{2\pi} \frac{1}{(5 \pm 4\sin t)^2}\, dt.$$

(c)

$$\int_0^{2\pi} \frac{\cos t}{5 \pm 4\sin t}\, dt.$$

(d)

$$\int_0^{2\pi} \frac{\cos t}{(5 \pm 4\sin t)^2}\, dt.$$

14. Repeat the previous exercise, with $5 \pm 4\sin t$ replaced by $5 \pm 4\cos t$.

15. Repeat it again, with $5 \pm 4\sin t$ replaced now by $25 \pm 6\cos t \pm 8\sin t$. Consider all four different combinations of signs.

Liouville's Theorem, argument principle, Rouché's Theorem.

16. Investigate our proof of the fundamental theorem of algebra using Liouville's Theorem and explain why it cannot be used to show that the function $f(z) = e^z$ must have a zero.

17. Determine how many zeroes the function $e^z + \sin z$ has on the region

$$\{(x, y) \mid x \in [1, 10],\ -\frac{1}{2}x \le y \le \frac{1}{2}x\}.$$

Boundary-value problems for Laplace's equation.
Solve the following problems.

18.

$$\Delta u = 0 \text{ on the lower half-plane}, \qquad u(x, 0) = \begin{cases} 0, & x \in (-\infty, -1) \\ 1 - \frac{1}{\pi}\arccos x, & x \in (-1, 1) \\ 1, & x \in (1, \infty) \end{cases}$$

19. The previous exercise with 'lower' replaced by 'upper'. [Hint: the solution here is a trivial adaptation of that in the previous exercise.]

20.

$$\Delta u = 0 \text{ on the right half-plane } \{(x, y) \mid x > 0\}, \qquad u(0, y) = \begin{cases} 0, & y \in (-\infty, -1) \\ 1, & y \in (1, \infty) \end{cases},$$
$$\left.\frac{\partial u}{\partial x}\right|_{(0,y)} = 0 \text{ for } y \in (-1, 1).$$

21. The previous problem with 'right' replaced by 'left'. [Hint: as before, the solution here is a trivial adaptation of that in the previous exercise.]

22.

$$\Delta u = 0 \text{ on the wedge } \{(x, y) \mid x > 0,\ -x < y < x\}, \qquad u(x, \pm x) = \pm\frac{2x^2}{4x^4 + 1}.$$

[Hint: use a conformal transformation which is a power, together with the fact that the real and imaginary parts of the function $f(z) = 1/(z + 1)$ are harmonic on the right half-plane.]

23.

$$\Delta u = 0 \text{ in the first quadrant } \{(x, y) \mid x > 0,\ y > 0\}, \qquad u(x, 0) = -\frac{2x^2}{4x^4 + 1}, \quad u(0, y) = \frac{2y^2}{4y^4 + 1}.$$

[Hint: use a conformal transformation – a multiplication by a complex number! – to transform this problem to the previous one.]

24. Pick some analytic function on the domain of one of our standard conformal maps, determine its real and imaginary parts, and work out what harmonic functions those map to under the chosen conformal map. Repeat this exercise as much as you like.