MAT334, COMPLEX VARIABLES, SUMMER 2020. PRACTICE PROBLEMS FOR MAY 11 – 15

1. For each complex number z, integer m, and interval (a,b), compute the mth root of z corresponding to the unique polar representation with angle in (a,b):
(a) $z=1$, $m=3$, $(-\pi,\pi)$. [This one is easy!]
(b) $z=1$, $m=3$, $\left(\frac{\pi}{2},\frac{5\pi}{2}\right)$. [Hint: what value for θ in this interval will give you a positive real number?]
(c) $z=-1$, $m=2$, $(0,2\pi)$.
(d) $z=-1$, $m=2$, $(2\pi,4\pi)$.
(e) $z=-i$, $m=4$, $(-\pi,\pi)$.
(f) $z=-27$, $m=3$, $\left(-\frac{3\pi}{2},\frac{\pi}{2}\right)$.
(g) $z=64$, $m=6$, $(-\pi,\pi)$.
(h) $z=64$, $m=6$, $(3\pi,5\pi)$.

(a) We must write $z=1(\cos\theta+i\sin\theta)$ where $\theta\in(-\pi,\pi)$; thus clearly we must take $\theta=0$, whence the corresponding root is $1^{1/3}\left(\cos\frac{0}{3}+i\sin\frac{0}{3}\right)=1$.

(b) Again, we must write $z=1(\cos\theta+i\sin\theta)$ where $\theta\in\left(\frac{\pi}{2},\frac{5\pi}{2}\right)$; in this case, then, we must take $\theta=2\pi$, whence the corresponding root is $1^{1/3}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$.

(c) Here we must write $z=1(\cos\theta+i\sin\theta)$ where $\theta\in(0,2\pi)$; thus we take $\theta=\pi$, whence we obtain the root $1^{1/2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i$.

(d) We have $z=1(\cos 3\pi+i\sin 3\pi)$, $z^{1/m}=1^{1/2}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-i$.

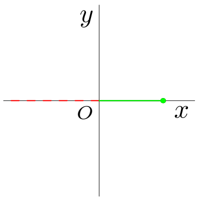

Fig. 1 *a*

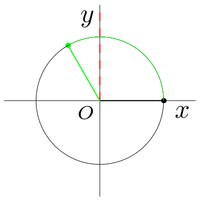

Fig. 1 *b*

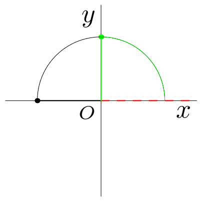

Fig. 1 *c*

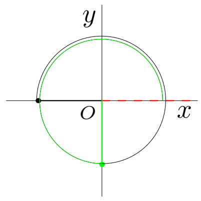

Fig. 1 *d*

(e) We must write $-i=1(\cos\theta+i\sin\theta)$, where $\theta\in(-\pi,\pi)$; evidently this requires $\theta=-\frac{\pi}{2}$, and the corresponding root is $1^{1/4}\left(\cos\left(-\frac{\pi}{8}\right)+i\sin\left(-\frac{\pi}{8}\right)\right)=\frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{1}{\sqrt{2}}}-i\sqrt{1-\frac{1}{\sqrt{2}}}\right)$.

(f) We have $z=27(\cos(-\pi)+i\sin(-\pi))$, $z^{1/m}=3\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)=\frac{3}{2}-i\frac{3\sqrt{3}}{2}$.

(g) We have as in (a) $z=64(\cos 0+i\sin 0)$, $z^{1/m}=2(\cos 0+i\sin 0)=2$.

(h) Here we must write $64=64(\cos\theta+i\sin\theta)$ for $\theta\in(3\pi,5\pi)$, which means we must take $\theta=4\pi$ (since $\cos\theta+i\sin\theta=1$ only when θ is a multiple of 2π, and 4π is the only multiple of 2π in that interval); thus we have $z^{1/m}=2\left(\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}\right)=-1+i\sqrt{3}$.

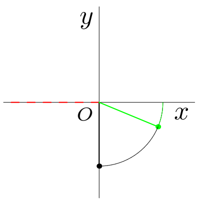

Fig. 1 *e*

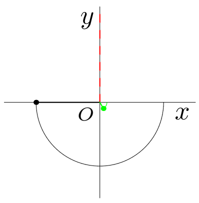

Fig. 1 *f*

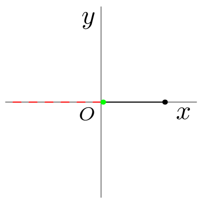

Fig. 1 *g*

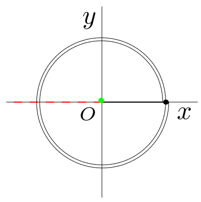

Fig. 1 *h*

2. Consider the mth root function restricted to the θ interval $(\theta_0,\theta_0+2\pi)$. Find the differences

$$\lim_{\theta\to\theta_0+2\pi^-} z-\lim_{\theta\to\theta_0^+} z,$$
$$\lim_{\theta\to\theta_0+2\pi^-} z^{1/m}-\lim_{\theta\to\theta_0^+} z^{1/m},$$

where z is a complex number with fixed modulus and argument θ. Explain what this means in terms of our ability to find a continuous mth root function on the entire complex plane.

We let $z = r(\cos\theta + i\sin\theta)$ and observe that

$$\lim_{\theta\to\theta_0+2\pi^-} z = r(\cos\theta_0 + i\sin\theta_0),$$
$$\lim_{\theta\to 0^+} z = r(\cos\theta_0 + i\sin\theta_0),$$

so that the first difference is 0, while

$$\lim_{\theta\to\theta_0+2\pi^-} z^{1/m} = \lim_{\theta\to\theta_0+2\pi^-} r^{1/m}\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right) = r^{1/m}\left(\cos\frac{\theta_0+2\pi}{m} + i\sin\frac{\theta_0+2\pi}{m}\right),$$
$$\lim_{\theta\to\theta_0^+} z^{1/m} = \lim_{\theta\to\theta_0^+} r^{1/m}\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right) = r^{1/m}\left(\cos\frac{\theta_0}{m} + i\sin\frac{\theta_0}{m}\right),$$

meaning that the second difference is

$$r^{1/m}\left(\cos\frac{\theta_0+2\pi}{m} - \cos\frac{\theta_0}{m} + i\left[\sin\frac{\theta_0+2\pi}{m} - \sin\frac{\theta_0}{m}\right]\right),$$

which will not be zero unless $m = 1$. This is just a very concrete demonstration of the fact, discussed in detail in the lecture notes, that any particular choice of mth root must have a jump discontinuity at some point if we attempt to extend it to the entire complex plane.

3. Suppose that a certain complex number z is given by $z = r(\cos\theta + i\sin\theta)$. Let $z^{1/m}$ denote the mth root of z corresponding to this particular polar representation. By how much must we increase θ to get a polar representation which gives the same mth root?

Since the mth root corresponding to this particular representation is given by

$$z^{1/m} = r^{1/m}\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right),$$

it is clearly necessary to increase θ by some integer multiple of $2\pi m$ in order to arrive at a polar representation giving the same mth root. (Note that it is only necessary to increase θ by some integer multiple of 2π itself in order to get another polar representation of z; in other words, to get a polar representation which also gives the same mth root, we must go around m times as far.)

4. Show that the function $P(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic in two different ways: (a) by direct computation; (b) by finding an analytic function of which it is the real part.

(a) We have

$$\frac{\partial P}{\partial x} = 4x^3 - 12xy^2, \qquad \frac{\partial^2 P}{\partial x^2} = 12x^2 - 12y^2,$$
$$\frac{\partial P}{\partial y} = -12x^2y + 4y^3, \qquad \frac{\partial^2 P}{\partial y^2} = -12x^2 + 12y^2,$$
$$\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 12x^2 - 12y^2 + (-12x^2 + 12y^2) = 0.$$

(b) This question was phrased slightly ambiguously. Evidently what the author had in mind was to recognise this function P as the real part of some known analytic function. It looks like it might be the real part of z^4. We can check this as follows:

$$(x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3),$$

and since we know that z^4 is an analytic function of z, this implies that $P(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic.

5. Do the same thing for the function $P(x, y) = 3x^2y - y^3$.

(a) Again, we have

$$\frac{\partial P}{\partial x} = 6xy, \qquad \frac{\partial^2 P}{\partial x^2} = 6y,$$
$$\frac{\partial P}{\partial y} = 3x^2 - 3y^2, \qquad \frac{\partial^2 P}{\partial y^2} + -6y,$$
$$\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 6y + (-6y) = 0.$$

(b) This function looks like it might be related to z^3. We have

$$(x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3),$$

so that $P(x,y) = 3x^2y - y^3$ is harmonic as it is the imaginary part of the analytic function z^3. Alternatively, it is the real part of the analytic function $-iz^3$.

6. The *Laplacian* $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, when written in polar coordinates, is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(a) Use this to show that the function $P(r,\theta) = \log r$ is harmonic on the set $\{(r,\theta) | r \neq 0\}$ (this is known as the *punctured plane* and is just the complex plane without the origin).

We have

$$\frac{\partial P}{\partial r} = \frac{1}{r}, \qquad \frac{\partial^2 P}{\partial r^2} = -\frac{1}{r^2}, \qquad \frac{\partial P}{\partial \theta} = 0,$$
$$\Delta P = -\frac{1}{r^2} + \frac{1}{r}\cdot\frac{1}{r} = 0,$$

where the calculations are valid as long as $r \neq 0$.

By our work from class, we know that there must, at least on some appropriate region of the plane, be another harmonic function Q such that $P + iQ$ is analytic, and that (see §3 of Goursat, or §9 of the lecture notes) Q is given by

$$Q(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\,dx + \frac{\partial P}{\partial x}\,dy.$$

(b) Let $(x_0, y_0) = (1, 0)$. By expressing P in terms of rectangular coordinates (x, y), evaluate the above integral when (i) $(x,y) = (0,1)$, (ii) $(x,y) = (0,-1)$, (iii) $(x,y) = (\cos\theta, \sin\theta)$, $\theta \in (0, 2\pi)$.

Since $r = \sqrt{x^2+y^2}$, we have $P = \frac{1}{2}\log(x^2+y^2)$, whence

$$\frac{\partial P}{\partial x} = \frac{x}{x^2+y^2}, \qquad \frac{\partial P}{\partial y} = \frac{y}{x^2+y^2}.$$

We note that cases (i) and (ii) are actually special cases of (iii), with $\theta = \pi/2$ and $\theta = \pi$, respectively, so we shall only show how to do that last case. Note that the initial point $(x_0, y_0) = (1, 0)$ and the final point (x, y) both lie on the unit circle. Thus we shall use the unit circle as the curve joining the two points. (This requires saying something about the path dependence of the integral; we shall address this point below so as not to interrupt the computational side of the question.) The unit circle can be parameterised as

$$(\cos t, \sin t), \qquad t \in [0, 2\pi],$$

and thus we may write

$$\begin{aligned} Q(\cos\theta, \sin\theta) &= \int_{(1,0)}^{(\cos\theta,\sin\theta)} -\frac{\partial P}{\partial y}\,dx + \frac{\partial P}{\partial x}\,dy \\ &= \int_0^\theta -\frac{\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t)\,dt \\ &= \int_0^\theta \sin^2 t + \cos^2 t\,dt = \theta. \end{aligned}$$

In particular, we have $Q(0,1) = \pi/2$ and $Q(0,-1) = \pi$.

In order to determine whether the above integral depends on the path taken, we need, by Green's theorem,[1] to determine whether any two such paths both lie in a single simply-connected region on which P is harmonic. The easiest way to ensure this, which is permissible since we are only interested in values of θ lying in the interval $(0, 2\pi)$, is to consider P and Q as defined only on the complex plane without the origin and the positive real axis: this region is clearly simply connected, and so any two paths must give the same value for the integral defining Q. The strong resemblance this procedure bears to taking a branch cut is by no means a coincidence.

(To be precise, the original integral definition of Q should have included also the stipulation that the paths involved were to be taken in this cut plane. Without such a stipulation, Q as given would not be well-defined. This point evidently slightly escaped the author when he was constructing this question.)

(c) Using your result from (b)(iii), what is

$$\lim_{\theta \to 2\pi^-} Q(\cos\theta, \sin\theta)?$$

What is

$$\lim_{\theta \to 0^+} Q(\cos\theta, \sin\theta)?$$

From this, does it appear that we can define Q continuously over the entire punctured plane? If not, does this contradict Green's theorem or what we know about conservative vector fields? Why or why not?

Clearly, we have

$$\begin{aligned} \lim_{\theta \to 2\pi^-} Q(\cos\theta, \sin\theta) &= \lim_{\theta \to 2\pi^-} \theta = 2\pi, \\ \lim_{\theta \to 0^+} Q(\cos\theta, \sin\theta) &= \lim_{\theta \to 0^+} \theta = 0. \end{aligned}$$

Since the point $(\cos\theta, \sin\theta)$ goes to the same point in the complex plane in both cases, it is evidently impossible to define Q continuously over the entire punctured plane. This does not contradict Green's theorem or what we know about conservative vector fields though since the punctured plane is not simply connected.

Let us go over all this a little bit more carefully. We have the integral definition

$$Q(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\,dx + \frac{\partial P}{\partial x}\,dy.$$

For this definition to make sense, though, the integral cannot depend on the path taken from (x_0, y_0) to (x, y), or otherwise Q would be a function on paths rather than a function of the endpoint (x, y) alone. At the risk of repeating some background from multivariable calculus, let us go over how to determine whether this integral is in fact path-independent. Suppose that γ_1 and γ_2 are two paths from (x_0, y_0) to (x, y); we may assume that they are both parameterised on the interval $[0, 1]$. For simplicity we assume that they do not intersect away from the endpoints; then the curve we obtain by running γ_1 forwards and γ_2 backwards, i.e., the curve parameterised on $[0, 2]$ by

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [0,1] \\ \gamma_2(t-1), & t \in [1,2] \end{cases},$$

is a closed curve, and as long as it does not contain the origin we may apply Green's theorem to conclude that

$$\begin{aligned} \int_\gamma -\frac{\partial P}{\partial y}\,dx + \frac{\partial P}{\partial x}\,dy &= \iint_D \frac{\partial}{\partial x}\frac{\partial P}{\partial x} - \frac{\partial}{\partial y}\left(-\frac{\partial P}{\partial y}\right)\,dA \\ &= \iint_D \Delta P\,dA = 0. \end{aligned}$$

[1] Note that the quantity inside the area integral in Green's theorem in this case is simply $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \Delta P = 0$, by (a).

If γ does enclose the origin, Green's theorem cannot be applied, since P is not defined there. Since it is not hard to show that

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{x} = \int_{\gamma_1} \mathbf{F} \cdot d\mathbf{x} - \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{x}$$

for any vector field $\mathbf{F}$ integrable over γ, this shows that the integral defining Q will be equal along any two paths which do not enclose the origin between them. If we require all paths to lie in the complex plane with the positive real axis and origin removed, then no two paths can enclose the origin, so with this restriction Q will be well-defined. However, since we have removed the positive real axis and the origin (note that when we say the paths must lie in a certain set, that condition must apply to their endpoints as well!), we must restrict the definition of Q in order to do this. Note that the limits in (c) correspond to two different curves from the point $(1, 0)$ to itself, namely the constant path $(x(t), y(t)) = (1, 0)$ and the full unit circle $(x(t), y(t)) = (\cos t, \sin t)$, which *do* enclose the origin, and this explains why the two limits above are not equal. (This particular example is actually closely related to the residue calculus which we shall study carefully and use extensively later on in the course.)

[At the risk of pointing out the obvious, the analytic function $f(x+iy) = P(x, y) + iQ(x, y)$, at least on the plane with the positive real axis and origin removed, is the complex logarithm we discussed in lecture on Thursday.]

7. (a) Show that the function $P(x, y) = e^x \cos y$ is harmonic on the entire plane.
This is straightforward:

$$\frac{\partial P}{\partial x} = e^x \cos y, \qquad \frac{\partial^2 P}{\partial x^2} = e^x \cos y,$$
$$\frac{\partial P}{\partial y} = -e^x \sin y, \qquad \frac{\partial^2 P}{\partial y^2} = -e^x \cos y,$$
$$\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = e^x \cos y + (-e^x \cos y) = 0.$$

(b) Using the method discussed in §9 of the lecture notes and §3 of Goursat, find a harmonic function $Q(x, y)$ such that

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is analytic on the entire plane. [Note: you need to do this problem using the indicated method, not by simply giving the answer!]

The relevant formula is that given in problem 6 above:

$$Q(x, y) = \int_{(x_0, y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy.$$

In this case – unlike there – the function P is harmonic on the entire plane, so that the integral above is guaranteed to be path-independent with no modifications of the domain of P or Q. Here it is convenient to integrate first along a path parallel to the x-axis and then along a path parallel to the y-axis, i.e., along the path

$$\gamma(t) = \begin{cases} (x_0 + (x - x_0)t, y_0), & t \in [0, 1] \\ (x, y_0 + (y - y_0)(t - 1)), & t \in [1, 2] \end{cases},$$

which gives (since $y'(t) = 0$ on the first part and $x'(t) = 0$ on the second), writing carefully (x, y) for the endpoint of the curve and $(x(t), y(t))$ for an arbitrary point along it,

$$\begin{aligned} Q(x, y) &= \int_0^1 e^{x(t)} \sin y_0\, (x - x_0)\, dt + \int_1^2 e^x \cos y(t)\, (y - y_0)\, dt \\ &= \sin y_0 \int_0^1 (x - x_0) e^{x_0} e^{(x-x_0)t}\, dt + e^x \int_1^2 \cos[y_0 + (y - y_0)(t - 1)](y - y_0)\, dt \\ &= e^{x_0} \sin y_0\, e^{(x-x_0)t}\Big|_0^1 + e^x \sin[y_0 + (y - y_0)(t - 1)]\big|_1^2 \\ &= e^{x_0} \sin y_0 \left(e^{x-x_0} - 1\right) + e^x (\sin y - \sin y_0) \\ &= e^x \sin y_0 - e^{x_0} \sin y_0 + e^x \sin y - e^x \sin y_0 = e^x \sin y - e^{x_0} \sin y_0. \end{aligned}$$

Thus, up to an additive constant, we may take $Q = e^x \sin y$, whence the function f becomes

$$f(x+iy) = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^{x+iy},$$

i.e., $f(z) = e^z$.

8.* [This is a more general version of problem 6 above.] (a) Show that the function

$$P(x,y) = \frac{1}{2}\log(x^2+y^2)$$

is harmonic on the punctured plane $\{(x,y)|(x,y) \neq (0,0)\}$.

Except for the 1/2, this is really just the function we saw in problem 6, except in rectangular coordinates. It is instructive to compute the Laplacian of this function in rectangular coordinates for practice though:

$$\begin{aligned}
\frac{\partial P}{\partial x} &= \frac{x}{x^2+y^2}, & \frac{\partial^2 P}{\partial x^2} &= \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\
\frac{\partial P}{\partial y} &= \frac{y}{x^2+y^2}, & \frac{\partial^2 P}{\partial y^2} &= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2},
\end{aligned}$$

$$\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0.$$

(b) Find a harmonic function $Q(x,y)$ on the *upper half-plane* $\{(x,y)|y>0\}$ such that the function

$$f(x+iy) = P(x,y) + iQ(x,y)$$

is analytic there. [Hint: pick $(x_0, y_0) = (1,0)$ as in problem 5, and then integrate along a circle and a portion of a straight line from the origin.]

Problem 5 should have been problem 6, somehow the numbering didn't get corrected. The idea is to do something similar to what we did in problem 7, except since we are polar coordinates here we will integrate along circles and rays directed radially from the origin, i.e., curves of constant θ and r, instead of lines parallel to the coordinate axes, which were curves of constant y and x. Let (x,y) be any point in the upper half-plane. Then we can write $x = r'\cos\theta'$ and $y = r'\sin\theta'$ where $\theta' \in (0,\pi)$. Let us define our curve γ as follows:

$$\gamma(t) = \begin{cases} (\cos\theta' t, \sin\theta' t), & t \in [0,1] \\ (1+(r'-1)(t-1)\cos\theta', 1+(r'-1)(t-1)\sin\theta'), & t \in [1,2] \end{cases}.$$

We note that in polar coordinates, at a point with polar coordinates (r,θ),

$$\frac{\partial P}{\partial x} = \frac{1}{r}\cos\theta, \qquad \frac{\partial P}{\partial y} = \frac{1}{r}\sin\theta,$$

so the integral for Q becomes

$$\begin{aligned}
Q(x,y) &= \int_0^1 \left(-\sin\theta' t(-\theta'\sin\theta' t) + \cos\theta' t(\theta'\cos\theta' t)\right) dt \\
&\quad + \int_1^2 \left(-\frac{1}{1+(r'-1)(t-1)}\sin\theta'\left[-\frac{r'-1}{1+(r'-1)(t-1)}\cos\theta'\right]\right. \\
&\qquad \left. + \frac{1}{1+(r'-1)(t-1)}\cos\theta'\left[-\frac{r'-1}{1+(r'-1)(t-1)}\sin\theta'\right]\right) dt \\
&= \int_0^1 \theta'\, dt + \int_1^2 0\, dt = \theta'.
\end{aligned}$$

(It would probably be simpler to integrate along the real axis first and then along a circular arc of radius r'; we leave the calculations in this case to the reader.)

(c) Repeat (b) for the *lower half-plane* $\{(x,y)|y<0\}$.

In this case we shall integrate clockwise, and make use of the suggestion at the end of the solution to (c) just now. Thus we shall write $(x,y)=(r'\cos\theta', r'\sin\theta')$, where we take $\theta'\in(-\pi,0)$, and use the path

$$\gamma(t)=\begin{cases}(1+(r'-1)t,0), & t\in[0,1]\\ (r'\cos\theta'(t-1), r'\sin\theta'(t-1)), & t\in[1,2]\end{cases}.$$

On the first segment we have $y'(t)=0$ and $\frac{\partial P}{\partial y}=0$, so the integral along that portion in the definition of Q vanishes; thus Q is simply the integral along the second segment:

$$\begin{aligned}Q(x,y)&=\int_1^2\left(-\frac{1}{r'}\sin\theta'(t-1)\left[-r'\theta'\cos\theta'(t-1)\right]+\frac{1}{r'}\cos\theta'(t-1)\left[r'\theta'\cos\theta'(t-1)\right]\right)\,dt\\ &=\int_1^2\theta'\,dt=\theta',\end{aligned}$$

formally the same as we found before, but remember that now $\theta'\in(0,\pi)$.

(d) Is it possible to find a function $Q(x,y)$ which is harmonic on the punctured plane and such that

$$f(x+iy)=P(x,y)+iQ(x,y)$$

is analytic everywhere on the punctured plane?

If this were possible, the function Q would have to agree on the lower and upper half-planes with the results of (b) and (c) above, up to additive constants; in other words, there would be constants C_1 and C_2 such that

$$Q(r\cos\theta, r\sin\theta)=\begin{cases}\theta+C_1, & \theta\in(0,\pi)\\ \theta+C_2, & \theta\in(-\pi,0)\end{cases}.$$

For Q to be continuous, we would need to have

$$\begin{aligned}\lim_{\theta\to0^-}Q(r\cos\theta,r\sin\theta)&=C_2\\ &=\lim_{\theta\to0^+}Q(r\cos\theta,r\sin\theta)=C_1,\\ \lim_{\theta\to\pi^-}Q(r\cos\theta,r\sin\theta)&=\pi+C_1\\ &=\lim_{\theta\to-\pi^+}Q(r\cos\theta,r\sin\theta)=C_2-\pi,\end{aligned}$$

so that we have the system of equations

$$\begin{aligned}C_1&=C_2\\ C_1+\pi&=C_2-\pi\end{aligned}$$

which clearly has no solution. This it is not possible to find such a function Q.

(e) How does your result from (d) relate to what we know about conservative vector fields and Green's theorem? Is there any contradiction?

The issues here are essentially the same as those treated in 6(c) above and we shall pass over them here.