MAT334, COMPLEX VARIABLES, SUMMER 2020. PRACTICE PROBLEMS FOR MAY 11 – 15

1. For each complex number z, integer m, and interval (a, b), compute the mth root of z corresponding to the unique polar representation with angle in (a, b):
(a) $z = 1$, $m = 3$, $(-\pi, \pi)$. [This one is easy!]
(b) $z = 1$, $m = 3$, $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$. [Hint: what value for θ in this interval will give you a positive real number?]
(c) $z = -1$, $m = 2$, $(0, 2\pi)$.
(d) $z = -1$, $m = 2$, $(2\pi, 4\pi)$.
(e) $z = -i$, $m = 4$, $(-\pi, \pi)$.
(f) $z = -27$, $m = 3$, $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$.
(g) $z = 64$, $m = 6$, $(-\pi, \pi)$.
(h) $z = 64$, $m = 6$, $(3\pi, 5\pi)$.

2. Consider the mth root function restricted to the θ interval $(\theta_0, \theta_0 + 2\pi)$. Find the differences

$$\lim_{\theta\to\theta_0+2\pi^-} z - \lim_{\theta\to\theta_0^+} z,$$
$$\lim_{\theta\to\theta_0+2\pi^-} z^{1/m} - \lim_{\theta\to\theta_0^+} z^{1/m},$$

where z is a complex number with fixed modulus and argument θ. Explain what this means in terms of our ability to find a continuous mth root function on the entire complex plane.

3. Suppose that a certain complex number z is given by $z = r(\cos\theta + i\sin\theta)$. Let $z^{1/m}$ denote the mth root of z corresponding to this particular polar representation. By how much must we increase θ to get a polar representation which gives the same mth root?

4. Show that the function $P(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic in two different ways: (a) by direct computation; (b) by finding an analytic function of which it is the real part.

5. Do the same thing for the function $P(x, y) = 3x^2y - y^3$.

6. The *Laplacian* $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, when written in polar coordinates, is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(a) Use this to show that the function $P(r, \theta) = \log r$ is harmonic on the set $\{(r, \theta) | r \neq 0\}$ (this is known as the *punctured plane* and is just the complex plane without the origin).

By our work from class, we know that there must, at least on some appropriate region of the plane, be another harmonic function Q such that $P + iQ$ is analytic, and that (see §3 of Goursat, or §9 of the lecture notes) Q is given by

$$Q(x, y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\,dx + \frac{\partial P}{\partial x}\,dy.$$

(b) Let $(x_0, y_0) = (1, 0)$. By expressing P in terms of rectangular coordinates (x, y), evaluate the above integral when (i) $(x, y) = (0, 1)$, (ii) $(x, y) = (0, -1)$, (iii) $(x, y) = (\cos\theta, \sin\theta)$, $\theta \in (0, 2\pi)$.

(c) Using your result from (b)(iii), what is

$$\lim_{\theta\to 2\pi^-} Q(\cos\theta, \sin\theta)?$$

What is

$$\lim_{\theta\to 0^+} Q(\cos\theta, \sin\theta)?$$

From this, does it appear that we can define Q continuously over the entire punctured plane? If not, does this contradict Green's theorem or what we know about conservative vector fields? Why or why not?

7. (a) Show that the function $P(x, y) = e^x \cos y$ is harmonic on the entire plane.

(b) Using the method discussed in §9 of the lecture notes and §3 of Goursat, find a harmonic function $Q(x, y)$ such that

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is analytic on the entire plane. [Note: you need to do this problem using the indicated method, not by simply giving the answer!]

8.* [This is a more general version of problem 6 above.] (a) Show that the function

$$P(x, y) = \frac{1}{2}\log(x^2 + y^2)$$

is harmonic on the punctured plane $\{(x, y)|(x, y) \neq (0, 0)\}$.

(b) Find a harmonic function $Q(x, y)$ on the *upper half-plane* $\{(x, y)|y > 0\}$ such that the function

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is analytic there. [Hint: pick $(x_0, y_0) = (1, 0)$ as in problem 5, and then integrate along a circle and a portion of a straight line from the origin.]

(c) Repeat (b) for the *lower half-plane* $\{(x, y)|y < 0\}$.

(d) Is it possible to find a function $Q(x, y)$ which is harmonic on the punctured plane and such that

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is analytic everywhere on the punctured plane?

(e) How does your result from (d) relate to what we know about conservative vector fields and Green's theorem? Is there any contradiction?