MAT334, COMPLEX VARIABLES, SUMMER 2020. PRACTICE PROBLEMS FOR MAY 4 – 8

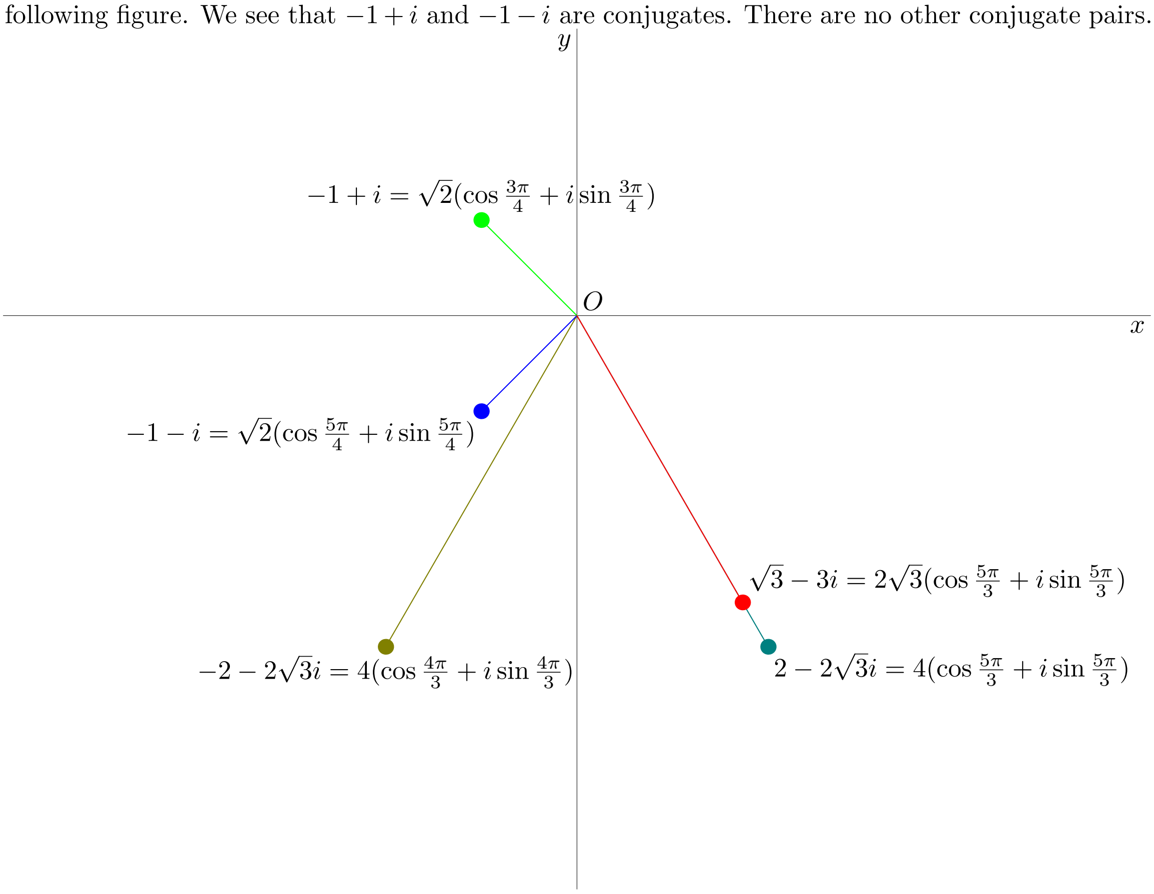
1. Perform the indicated arithmetic operations:

$$(2+3i)+(4-5i)=6-2i, \qquad (3-i)\cdot(4+2i)=12+2+i(6-4)=14+2i$$
$$(10-3i)\cdot(1-2i)=10-6+i(-20-3)=4-23i$$
$$(-1+3i)\cdot(1+4i)=-1-12+i(-4+3)=-13-i, \qquad (-3-3i)\cdot(-4+4i)=12+12+i(-12+12)=24$$
$$\frac{3-i}{4+2i}=\frac{(3-i)\cdot(4-2i)}{16+4}=\frac{12-2+i(-6-4)}{20}=\frac{1}{2}-\frac{1}{2}i$$
$$\frac{10-i}{3-4i}=\frac{(10-i)\cdot(3+4i)}{9+16}=\frac{30+4+i(40-3)}{25}=\frac{34}{25}+\frac{37}{25}i$$
$$\frac{-1+2i}{4-3i}=\frac{(-1+2i)\cdot(4+3i)}{16+9}=\frac{-4-6+i(-3+8)}{25}=-\frac{2}{5}+\frac{1}{5}i.$$

2. Plot the points corresponding to the following complex numbers on the complex plane. For each of them, find the modulus (length) of the complex number and its argument (the angle the corresponding point makes with the positive real axis), without using a calculator!

$$\sqrt{3}-3i, \qquad -1+i, \qquad -1-i, \qquad -2-2\sqrt{3}i, \qquad 2-2\sqrt{3}i.$$

See the following figure. We see that $-1+i$ and $-1-i$ are conjugates. There are no other conjugate pairs.



($-2-2\sqrt{3}i$ and $2-2\sqrt{3}i$ have opposite signs on their *real* parts, but I am not aware of any particular use for this kind of pairing.)

3. For each of the following complex numbers, find all roots of the indicated orders. (It is sufficient to write them in polar form with the answer in terms of sin and cos.) Plot the numbers and the corresponding roots on the complex plane.

$$-27i, 3; \qquad 16, 4; \qquad -\frac{125\sqrt{2}}{2}+\frac{125\sqrt{2}}{2}i, 3; \qquad \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i, 7.$$

[Hint: start out by finding the polar representation of each of these numbers (that is, write them as $r(\cos\theta + i\sin\theta)$).] The figures are as follows. The algebraic expressions are

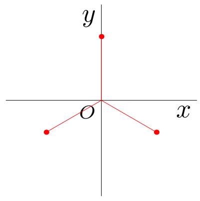


Fig. 3 *a*

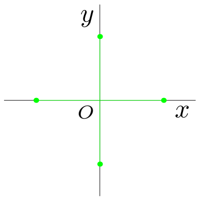


Fig. 3 *b*

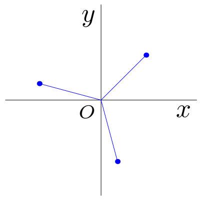


Fig. 3 *c*

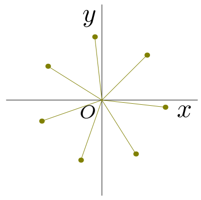


Fig. 3 *d*

$$-27i = 27\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right), \qquad 16 = 16\left(\cos 0 + i\sin 0\right),$$
$$-\frac{125\sqrt{2}}{2} + i\frac{125\sqrt{2}}{2} = 125\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \qquad \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4};$$

$$(-27i)^{1/3} = 3(\cos\left(\frac{\pi}{2} + \frac{2\pi n}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{2\pi n}{3}\right), \qquad n = 0, 1, 2$$
$$16^{1/4} = 2(\cos\frac{\pi n}{2} + i\sin\frac{\pi n}{2}), \qquad n = 0, 1, 2, 3$$
$$\left(-\frac{125\sqrt{2}}{2} + i\frac{125\sqrt{2}}{2}\right)^{1/3} = 5\left(\cos\left(\frac{\pi}{4} + \frac{2\pi n}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi n}{3}\right)\right), \qquad n = 0, 1, 2$$
$$\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)^{\frac{1}{7}} = \cos\left(\frac{\pi}{4} + \frac{2\pi n}{7}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi n}{7}\right), \qquad n = 0, 1, 2, 3, 4, 5, 6$$

4. Using the Cauchy-Riemann equations, determine which of the following functions are analytic at the indicated points. Make sure to show all of your work and justify your answers! [Recall that in addition to the Cauchy-Riemann equations, the partial derivatives must be continuous at the point in question, so you need to say something about that as well. But for the functions here it is pretty straightforward; we certainly don't expect you to prove anything using ϵ–δ arguments!]

$$f(x+iy) = x^3 - 3xy^2 + i(3x^2y - y^3), \qquad x + iy \text{ arbitrary}$$

We have

$$\frac{\partial}{\partial x}x^3 - 3xy^2 = 3x^2 - 3y^2, \qquad \frac{\partial}{\partial y}x^3 - 3xy^2 = -6xy$$
$$\frac{\partial}{\partial x}3x^2y - y^3 = 6xy, \qquad \frac{\partial}{\partial y}3x^2y - y^3 = 3x^2 - 3y^2$$

from which we see that the Cauchy-Riemann equations are satisfied, and that the partial derivatives are continuous, so that the function is analytic throughout the complex plane.

$$f(z) = z^4, \qquad z \text{ arbitrary}$$

Expanding, we have

$$z^4 = (x+iy)^4 = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3),$$

which gives

$$\frac{\partial}{\partial x}x^4 - 6x^2y^2 + y^4 = 4x^3 - 12xy^2, \qquad \frac{\partial}{\partial y}x^4 - 6x^2y^2 + y^4 = -12x^2y + 4y^3$$
$$\frac{\partial}{\partial x}4x^3y - 4xy^3 = 12x^2y - 4y^3, \qquad \frac{\partial}{\partial y}4x^3y - 4xy^3 = 4x^3 - 12xy^2$$

from which we see as before that the Cauchy-Riemann equations are satisfied, and that the first-order partials are continuous, so that this function is also analytic. (Another, perhaps easier, way of seeing this is indicated at the end of section 4 of the notes.)

$$f(x+iy) = e^x(\cos y + i \sin y), \qquad x+iy \text{ arbitrary}$$

In this case we have

$$\frac{\partial}{\partial x} e^x \cos y = e^x \cos y, \qquad \frac{\partial}{\partial y} e^x \cos y = -e^x \sin y$$
$$\frac{\partial}{\partial x} e^x \sin y = e^x \sin y, \qquad \frac{\partial}{\partial y} e^x \sin y = e^x \cos y$$

from which we again see that the Cauchy-Riemann equations are satisfied, and that the derivatives are continuous, so that the function is analytic.

$$f(x+iy) = 2xy - iy^2, \qquad x+iy \text{ arbitrary}$$

In this case we have

$$\frac{\partial}{\partial x} 2xy = 2y, \qquad \frac{\partial}{\partial y} 2xy = 2x$$
$$\frac{\partial}{\partial x} -y^2 = 0, \qquad \frac{\partial}{\partial y} -y^2 = -2y$$

so that, while the partial derivatives are continuous, neither of the Cauchy-Riemann equations are satisfied so the function is not analytic.

$$f(x+iy) = \cos x \cosh y - i \sin x \sinh y, \qquad x+iy \text{ arbitrary}$$

In this case we have

$$\frac{\partial}{\partial x} \cos x \cosh y = -\sin x \cosh y, \qquad \frac{\partial}{\partial y} \cos x \cosh y = \cos x \sinh y$$
$$\frac{\partial}{\partial x} -\sin x \sinh y = -\cos x \sinh y, \qquad \frac{\partial}{\partial y} -\sin x \sinh y = -\sin x \cosh y$$

from which we see that the Cauchy-Riemann equations are satisfied and that the partial derivatives are continuous, so that the function is analytic.

$$f(x+iy) = \sin x \cosh y - i \cos x \sinh y, \qquad x+iy \text{ arbitrary}$$

In this case, we have

$$\frac{\partial}{\partial x} \sin x \cosh y = \cos x \cosh y, \qquad \frac{\partial}{\partial y} \sin x \cosh y = \sin x \sinh y$$
$$\frac{\partial}{\partial x} -\cos x \sinh y = \sin x \sinh y, \qquad \frac{\partial}{\partial y} -\cos x \sinh y = -\cos x \cosh y$$

so that even though the partial derivatives are continuous, the Cauchy-Riemann equations are not satisfied so the function is not analytic. (Can you see how to change this function just a little bit in order to make it analytic?)

$$f(x+iy) = \sin x - i \cos y, \qquad x+iy \text{ arbitrary}$$

In this case we have

$$\frac{\partial}{\partial x} \sin x = \cos x, \qquad \frac{\partial}{\partial y} \sin x = 0$$
$$\frac{\partial}{\partial x} -\cos y = 0, \qquad \frac{\partial}{\partial y} -\cos y = \sin y$$

so that even though the partial derivatives are continuous and one of the Cauchy-Riemann equations is satisfied, the other one isn't, so the function is not analytic.

$$f(x+iy) = \left(x^2+y^2\right)^{\frac{1}{4}} \left(\sqrt{\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)} + i\sqrt{\frac{1}{2}\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)} \right), \qquad x+iy \text{ such that } x,y>0$$

This one is a bit more tricky. It is best to bring the leading factor inside the square root; doing this we obtain

$$\begin{aligned} \frac{\partial}{\partial x}\left(x^2+y^2\right)^{1/4}\sqrt{\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)} &= \frac{\partial}{\partial x}\sqrt{\frac{1}{2}\left(\sqrt{x^2+y^2}+x\right)} \\ &= \frac{1}{2\sqrt{2}}\frac{\frac{x}{\sqrt{x^2+y^2}}+1}{\sqrt{\sqrt{x^2+y^2}+x}}, \end{aligned}$$

from which it is fairly easy to see that we have also

$$\begin{aligned} \frac{\partial}{\partial y}\left(x^2+y^2\right)^{1/4}\sqrt{\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)} &= \frac{1}{2\sqrt{2}}\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}+x}}, \\ \frac{\partial}{\partial x}\left(x^2+y^2\right)^{1/4}\sqrt{\frac{1}{2}\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)} &= \frac{1}{2\sqrt{2}}\frac{\frac{x}{\sqrt{x^2+y^2}}-1}{\sqrt{\sqrt{x^2+y^2}-x}}, \\ \frac{\partial}{\partial y}\left(x^2+y^2\right)^{1/4}\sqrt{\frac{1}{2}\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)} &= \frac{1}{2\sqrt{2}}\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}-x}}. \end{aligned}$$

In order to compare these, we note that we may write

$$\begin{aligned} \frac{1}{2\sqrt{2}}\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}-x}}\cdot\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}+x} &= \frac{1}{2\sqrt{2}}\frac{y}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2-x^2}\cdot\sqrt{\sqrt{x^2+y^2}+x}} \\ &= \frac{1}{2\sqrt{2}}\frac{y}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2+y^2}+x}{y\sqrt{\sqrt{x^2+y^2}+x}} = \frac{1}{2\sqrt{2}}\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}+x}}, \end{aligned}$$

where we may write $\sqrt{y^2} = y$ since we have $y > 0$. This shows that the first of the Cauchy-Riemann equations is satisfied. The second may be shown similarly:

$$\begin{aligned} \frac{1}{2\sqrt{2}}\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}+x}}\cdot\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}-x} &= \frac{1}{2\sqrt{2}}\frac{y}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2-y^2}-x}{\sqrt{x^2+y^2-x^2}\cdot\sqrt{\sqrt{x^2+y^2}-x}} \\ &= \frac{1}{2\sqrt{2}}\frac{y}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2+y^2}-x}{y\sqrt{\sqrt{x^2+y^2}-x}} = \frac{1}{2\sqrt{2}}\frac{1-\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}-x}}, \end{aligned}$$

from which the second Cauchy-Riemann equation follows. Since the expressions above are moreover continuous when $x, y > 0$, we see that the function is in fact analytic on the first quadrant. [We also observe that our professor is unlikely to put so long a problem on the quiz, at least in full.]

$$f(x+iy) = x^3+3x^2y-3xy^2-y^3, \qquad x+iy \text{ arbitrary}$$

In this case we have

$$\frac{\partial}{\partial x}x^3+3x^2y-3xy^2-y^3=3x^2+6xy-3y^2, \qquad \frac{\partial}{\partial y}x^3+3x^2y-3xy^2-y^3=3x^2-6xy-3y^2$$
$$\frac{\partial}{\partial x}0=0, \qquad \frac{\partial}{\partial y}0=0$$

whence we see that though the partial derivatives are continuous, neither of the Cauchy-Riemann equations are satisfied, so that the function is not analytic. (This demonstrates, incidentally, that even functions which look very 'nice' (like polynomials!) when written out in terms of x and y very often to *not* give rise to analytic functions because the Cauchy-Riemann equations may not be satisfied.)

Consider the second-to-last function. What is its square?

$$\left[\left(x^2+y^2\right)^{\frac{1}{4}}\left(\sqrt{\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)}+i\sqrt{\frac{1}{2}\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)}\right)\right]^2$$
$$=\left(x^2+y^2\right)^{1/2}\left[\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)-\frac{1}{2}\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)+i\sqrt{\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)}\right]$$
$$=x+iy$$

where we must again use $x, y > 0$. Thus this function is the square root function restricted to the first quadrant.

5. Can a nonzero analytic function have an identically zero imaginary part (i.e., can it be entirely real)? an identically zero real part?

Let us consider a function

$$f(x+iy)=P(x,y)+iQ(x,y).$$

To say that this function has an identically zero imaginary part means that $Q(x,y)=0$ for all x and y. Were such a function analytic, the Cauchy-Riemann equations would give

$$\frac{\partial P}{\partial x}=0, \qquad \frac{\partial P}{\partial y}=0,$$

which means that P must be a constant. Similarly, the only analytic functions which have identically zero real part are constants ib where b is some real number.

6. [This problem verges a bit into what we will talk about next week.] Consider the function

$$P(x,y)=x^5-10x^3y^2+5xy^4.$$

Can you find a polynomial $Q(x,y)$ such that the function

$$f(x+iy)=P(x,y)+iQ(x,y)$$

is analytic at every point in the complex plane?

The point is that Q must satisfy the Cauchy-Riemann equations. These give

$$\frac{\partial Q}{\partial y}=\frac{\partial P}{\partial x}=5x^4-30x^2y^2+5y^4,$$
$$\frac{\partial Q}{\partial x}=-\frac{\partial P}{\partial y}=-20x^3y+20xy^3.$$

From the first of these, we see that any such Q must be of the form

$$Q(x,y)=5x^4y-10x^2y^3+y^5+q(x),$$

where q is some arbitrary function. Substituting this into the second equation, we find that

$$-20x^3y+20xy^3=20x^3y-20xy^3+q'(x),$$

so that $q'(x)=0$ and $q(x)=C$ for some constant C. Thus the function

$$f(x,y)=x^5-10x^3y^2+5xy^4+i(5x^4y-10x^2y^3+y^5+C)$$

will be analytic at every point in the complex plane. (Note that this is just $f(z)=z^5+iC$!)