MAT334, COMPLEX VARIABLES, SUMMER 2020. PRACTICE PROBLEMS FOR MAY 4 – 8

1. Perform the indicated arithmetic operations:

$$(2+3i)+(4-5i),\qquad (3-i)\cdot(4+2i),\qquad (10-3i)\cdot(1-2i),\qquad (-1+3i)\cdot(1+4i),\qquad (-3-3i)\cdot(-4+4i)$$
$$\frac{3-i}{4+2i},\qquad \frac{10-i}{3-4i},\qquad \frac{-1+2i}{4-3i}$$

2. Plot the points corresponding to the following complex numbers on the complex plane. For each of them, find the modulus (length) of the complex number and its argument (the angle the corresponding point makes with the positive real axis), without using a calculator!

$$\sqrt{3}-3i,\qquad -1+i,\qquad -1-i,\qquad -2-2\sqrt{3}i,\qquad 2-2\sqrt{3}i.$$

Are there any conjugate pairs in this list?

3. For each of the following complex numbers, find all roots of the indicated orders. (It is sufficient to write them in polar form with the answer in terms of sin and cos.) Plot the numbers and the corresponding roots on the complex plane.

$$-27i, 3;\qquad 16, 4;\qquad -\frac{125\sqrt{2}}{2}+\frac{125\sqrt{2}}{2}i, 3;\qquad \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i, 7.$$

[Hint: start out by finding the polar representation of each of these numbers (that is, write them as $r(\cos\theta + i\sin\theta)$).]

4. Using the Cauchy-Riemann equations, determine which of the following functions are analytic at the indicated points. Make sure to show all of your work and justify your answers! [Recall that in addition to the Cauchy-Riemann equations, the partial derivatives must be continuous at the point in question, so you need to say something about that as well. But for the functions here it is pretty straightforward; we certainly don't expect you to prove anything using ϵ–δ arguments!]

$$f(x+iy) = x^3 - 3xy^2 + i(3x^2y - y^3),\qquad x+iy \text{ arbitrary}$$
$$f(z) = z^4,\qquad z \text{ arbitrary}$$
$$f(x+iy) = e^x(\cos y + i\sin y),\qquad x+iy \text{ arbitrary}$$
$$f(x+iy) = 2xy - iy^2,\qquad x+iy \text{ arbitrary}$$
$$f(x+iy) = \cos x\cosh y - i\sin x\sinh y,\qquad x+iy \text{ arbitrary}$$
$$f(x+iy) = \sin x\cosh y - i\cos x\sinh y,\qquad x+iy \text{ arbitrary}$$
$$f(x+iy) = \sin x - i\cos y,\qquad x+iy \text{ arbitrary}$$
$$f(x+iy) = \left(x^2+y^2\right)^{\frac{1}{4}}\left(\sqrt{\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)}+i\sqrt{\frac{1}{2}\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)}\right),\qquad x+iy \text{ such that } x,y>0$$
$$f(x+iy) = x^3 + 3x^2y - 3xy^2 - y^3,\qquad x+iy \text{ arbitrary}$$

Consider the second-to-last function. What is its square?

5. Can a nonzero analytic function have an identically zero imaginary part (i.e., can it be entirely real)? an identically zero real part?

6. [This problem verges a bit into what we will talk about next week.] Consider the function

$$P(x,y) = x^5 - 10x^3y^2 + 5xy^4.$$

Can you find a polynomial $Q(x,y)$ such that the function

$$f(x+iy) = P(x,y) + iQ(x,y)$$

is analytic at every point in the complex plane?