APPENDIX I. REVIEW OF MULTIVARIABLE CALCULUS AND LINEAR ALGEBRA

I. MULTIVARIABLE CALCULUS

1. Parametric curves. A *plane parametric curve*[1] is a curve in the plane which can be described by two equations

$$x = x(t), \qquad y = y(t), \qquad\qquad (t \in [a, b])$$

for some interval $[a, b]$; in other words, for every point (x, y) on the curve, there is some value $t \in [a, b]$ such that $x = x(t)$ and $y = y(t)$. (Note that this t need not be unique.) More informally, if we view t as a dynamical quantity, the point $(x(t), y(t))$ 'traces out' the entire curve as t varies from a to b. It is often convenient to represent the point $(x(t), y(t))$ by a single function, often called $\gamma(t)$ (the Greek letter gamma), so that $\gamma(t) = (x(t), y(t))$. We shall use γ (without t) to refer to the entire curve, considered as a single object. When necessary to distinguish between the *function* $\gamma(t)$ and the plane curve this function represents, we shall call the latter the *image* of γ.

A curve is called *closed* when (in the notation of the previous paragraph) $\gamma(a) = \gamma(b)$. A closed curve which does not intersect itself (i.e., for which the value of t mentioned above *is* unique) is called a *Jordan curve*. A Jordan curve γ is said to be oriented *counterclockwise* if, as t increases from a to b, the point $\gamma(t)$ traces out the curve in a counterclockwise direction, and similarly to be oriented *clockwise* if this point traces out the curve in a clockwise direction.[2] We note for future use that if D is a connected region of the plane, then its boundary curve is always a Jordan curve. This result has a converse in the so-called *Jordan curve theorem* which we shall mention later on in the course.

General parametric curves can display pathological behaviour, even when $x(t)$ and $y(t)$ are both continuous.[3] In this course we shall deal exclusively with so-called *piecewise-smooth curves*, defined as follows. A parametric curve γ is said to be *piecewise-smooth* on an interval $[a, b]$ if (i) it is continuous on $[a, b]$ and (ii) there are points $t_0 = a < t_1 < \cdots < t_n = b$ such that on each subinterval (t_i, t_{i+1}), $i = 0, \cdots, n-1$, the *derivative* $\gamma'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$[4] exists, and is continuous and nonzero. (Condition (ii) amounts to saying that $x(t)$ and $y(t)$ are continuously differentiable on (t_i, t_{i+1}), and that $x'(t)$ and $y'(t)$ never vanish simultaneously. This last requirement is necessary to avoid 'corners'; see the practice problems!)

A piecewise smooth curve has a well-defined length. Recall that the *length* of a parametric curve γ defined on some interval $[a, b]$ and such that γ' is continuous there is given by

$$\int_a^b |\gamma'(t)|\, dt,$$

where $|\cdot|$ denotes the length of a vector. This definition can be extended to a piecewise-smooth curve in an obvious way: if $t_0, t_1, \ldots, t_n$ are the points given in the definition of piecewise-smoothness, then we define the length of γ to be[5]

$$\int_{t_0}^{t_1} |\gamma'(t)|\, dt + \int_{t_1}^{t_2} |\gamma'(t)|\, dt + \cdots + \int_{t_{n-1}}^{t_n} |\gamma'(t)|\, dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\gamma'(t)|\, dt.$$

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[1] Parametric curves can, of course, also be considered in three (and even arbitrary) dimensions. In this course, though, we shall only need them in two.

[2] Note that this definition would not make sense for a self-intersecting curve: for example, no matter how you trace out a figure-eight, the upper part will be oriented one way while the lower part will be oriented another.

[3] For example, one can find a parametric curve which – at least if we are allowed to replace the bounded interval $[a, b]$ by the whole real line – essentially fill out an entire two-dimensional region!

[4] While we shall not need to make this distinction in this course, it bears pointing out that, technically speaking, points and vectors are not identical, and when one must distinguish between them, a curve γ gives a point for each t while its derivative gives a vector.

[5] To be precise, the integrals here should be understood as improper integrals obtained by integrating from something slightly greater than t_i to something slightly less than t_{i+1}, and then taking the limit as these endpoints approach those two values, respectively; but this is generally not something which needs to be made explicit in practice, and we shall generally pass over it in silence in similar cases in the future.

The main use we shall make of parametric curves is in *line integrals* (see §3 below), and also in describing how two points in the plane are connected. This latter will become clearer as we progress through the course. The fact that two real numbers are essentially only connected in one way, while two complex numbers can be connected in multiple ways, some of which may be distinct (in an appropriate sense), is part of what makes complex analysis interesting.

2. Partial derivatives. Suppose that we have a function f defined on a region of the plane, which we suppose has Cartesian coordinates (x, y). We define its *partial derivatives* with respect to x and y to be

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$
$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{(f(x, y+h) - f(x, y)}{h}.$$

We recall that in multivariable calculus we saw that the *existence* of both partial derivatives still allowed for quite a bit of pathological behaviour. It turns out that for functions of a complex variable there are additional requirements on the partial derivatives in order for the function to have a single *complex* derivative, and that these requirements, though simple, lead to far-reaching results which rule out all such pathological behaviours.

Recall that if a function f has a local extremum at a point where its partial derivatives exist, then they must both vanish.

Some examples of partial derivatives are given in the review problems.

[This paragraph is an aside for students who have had MAT237 or MAT257, or who have otherwise learned how to view the derivative as a linear map. In this class we shall be interested in complex-valued functions of a complex variable; since the set of complex numbers is a two-dimensional vector space over the real numbers, this means that we are in essence considering functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ (or, in essence, a vector field on $\mathbf{R}^2$). Thus the derivative of such a function, in the multivariable-calculus sense, should be a linear map from $\mathbf{R}^2$ to $\mathbf{R}^2$ approximating the original function at the point of differentiation. It turns out that the requirement that a complex derivative exists requires that this map be a composition of an isotropic scaling (i.e., multiplication by a single real number) and a rotation. This is the basis for the study of functions of a complex variable as *conformal maps*, namely functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ which preserve angles.]

3. Line integrals and vector fields. Suppose that γ is a piecewise-smooth curve on an interval $[a, b]$ ($\gamma(t) = (x(t), y(t))$), and that f is a continuous function defined on some set containing the image of γ. Then we define three different types of *line integral* along γ, as follows. Let $t_0, t_1, \cdots, t_n$ be the points given in the definition of piecewise-smoothness; then we define

$$\int_\gamma f\, dx = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x(t), y(t))\, x'(t)\, dt$$
$$\int_\gamma f\, dy = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x(t), y(t))\, y'(t)\, dt$$
$$\int_\gamma f\, ds = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x(t), y(t))\, |\gamma'(t)|\, dt,$$

and call these *the line integrals of f along γ with respect to x, y, and arclength*, respectively.

Recall that a *vector field* on a region of $\mathbf{R}^2$ is a function which to every point in its domain associates a vector in $\mathbf{R}^2$; in other words, it can be written as a function $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, where $P(x, y)$ and $Q(x, y)$ are functions defined on the region called (naturally) the *components* of the vector field. If the vector field $\mathbf{F}$ is defined on a region containing γ, then we may combine the line integrals with respect to x and y of the components of $\mathbf{F}$ to define a new line integral, as follows:

$$\int_\gamma P(x, y)\, dx + \int_\gamma Q(x, y)\, dy = \int_\gamma \mathbf{F}(x, y) \cdot d\mathbf{x}.$$

We call this the line integral of the vector field along the curve γ. Recall the following *fundamental theorem of calculus for line integrals*: If $\mathbf{F} = \nabla f$ for some function f, i.e., if $\mathbf{F}$ is a gradient, then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = f(\gamma(b)) - f(\gamma(a)),$$

and this integral is therefore independent of the choice of path γ. This notion of *path-independence* (this is a standard term, though in our current setting it would be more natural to call it *curve-independence*!), namely that the line integral along a certain curve only depends on the end-points of the curve and not on the curve itself, is of central importance in the study of analytic functions of a complex variable. Recall that it is equivalent to the requirement that the line integral along any closed curve be zero. This is in turn related to *Green's theorem*, which states that for any vector field $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and any closed curve γ bounding a connected region D and oriented counterclockwise,

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where the latter is an *area integral* over the region D. This is a special case of *Stokes's theorem*, which we shall not need in its full generality but which we state here because it provides useful notation: If S is any (sufficiently smooth) connected surface in $\mathbf{R}^3$ with boundary curve C, and S and C are oriented consistently,[6] then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dA.$$

Here the second integral is a *surface integral* and $\mathbf{n}$ represents the *unit normal* to the surface S, but we shall not need these things in this class. The *curl* of a vector field can be defined heuristically as $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$; if $\mathbf{F}$ is a vector field on $\mathbf{R}^2$ then the curl can be taken to be the single number

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

appearing in Green's theorem. For us this is the only case for which we shall need to use the curl (and we shall not need to use it much even here).

Note now that Green's theorem tells us that line integrals of a vector field are path-independent exactly when the curl of that vector field is zero. Such a vector field is called *conservative*, though we shall only need this term only occasionally. We have seen that a vector field which is the gradient of a function is conservative; on a so-called *simply connected region* – by which we mean a region 'without holes', or, more precisely, whose boundary is a *single* Jordan curve – the converse is also true. We shall see that these results have analogues in the theory of functions of a complex variable, though the results generally are not quite exact copies.

II. LINEAR ALGEBRA

4. Matrices. In this course we shall not need much from the results of linear algebra, but mostly a familiarity with its concepts. Recall that a *matrix of size m by n* is a two-dimensional array of numbers

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

and is called *square* if $m = n$. The *product* of matrices $[a_{ij}]$ and $[b_{jk}]$ of sizes m by n and n by ℓ is defined to be the matrix $[c_{ik}]$ of size m by ℓ given by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

[6] For us, this just means that were γ oriented *clockwise* we would need to introduce an extra minus sign on the right-hand side of Green's theorem.

Recall that the *identity matrix of size n by n*

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

has the property that $AI = A$ and $IB = B$ for any matrices A and B of size m by n and n by ℓ, respectively. If a matrix $A = [a_{ij}]$ is square of size n by n, then its *inverse* (when it exists) is a matrix A^{-1} of size n by n satisfying

$$AA^{-1} = A^{-1}A = I.$$

In general, finding an inverse matrix is hard. For two-by-two matrices, however, there is a simple formula which is often useful, given by *Cramer's rule*: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

as long as $ad - bc \neq 0$. The quantity $ad - bc$ is called the *determinant* of the matrix A; the notion of determinant can be defined for a square matrix of any size, but as the general definition is complicated and we shall not need it in this course we pass over it for the moment.

Recall that a matrix of size m by n can be viewed as giving a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$. In particular, a 2 by 2 matrix can be viewed as a linear transformation on the plane. Two particularly important and simple examples are *isotropic scaling* and *rotation.* The first is just multiplication by a single scalar and corresponds to the matrix ($\lambda \neq 0$)

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad \text{which has inverse} \qquad \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}.$$

The second is a bit more complicated. Consider rotation of the plane by an angle θ counterclockwise around the origin. Since vector addition and scalar multiplication in the plane can be defined in terms of geometric pictures which are transformed rigidly by such a rotation, we see that this rotation must be linear; thus it suffices to determine its effect on the basis vectors $\mathbf{i}$ and $\mathbf{j}$ of the plane. If we rotate the vector $\mathbf{i}$ by an angle θ counterclockwise around the origin, a little geometry makes it clear that we obtain the vector $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, while if we rotate $\mathbf{j}$ the same way we obtain the vector $-\sin\theta i + \cos\theta\mathbf{j}$; thus the matrix giving this transformation is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We note two interesting properties of this matrix: first, its determinant is

$$\cos\theta \cdot \cos\theta - (-\sin\theta) \cdot \sin\theta = \cos^2\theta + \sin^2\theta = 1;$$

secondly, its inverse is (by the general formula above)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

which is just the original matrix with θ replaced by $-\theta$! This makes good sense since the inverse to a counterclockwise rotation by θ is a clockwise rotation by θ, which is essentially just a counterclockwise rotation by $-\theta$.